Math 421/510 Homework 3: Due on Friday, Feb. 16, 11AM

1. Let μ be Lebesgue measure on $[0, 1]$.

   (a) Show that if $f, g \in C([0, 1])$ and $f = g$ μ-a.e., then $f = g$, and thus for any $1 \leq p \leq \infty$, $C([0, 1])$ may be regarded as a subspace of $L^p([0, 1], \mu)$.

   (b) Show that the sup norm and L^1 norm on $C([0, 1])$ are not equivalent.

   (c) Are the sup norm and L^∞ norm on $C([0, 1])$ equivalent?

   Solution:

   a. If $f = g$ μ-a.e., then they must agree on at least point in every subinteral and so they agree on a dense set. . Since they are both continuous, they are equal.

   b. For $f(x) = 1 - nx$ on $[0, 1/n]$ and 0 elsewhere, $||f||_{\sup} = 1, ||f||_1 = 1/2n$ and so $\frac{||f||_{\sup}}{||f||_1} = 2n$ can be arbitrarily large.

   Alternatively note that w.r.t. the sup norm $C([0, 1])$ is complete but w.r.t the L^1 norm is not complete.

   c. Yes. We claim that the sup and essential sup of a continuous function are equal. To see this, observe that for any $y < \sup f$, $f^{-1}(y, \infty)$ is nonempty and open and thus contains a non-empty open interval and thus has positive measure. Thus, $\sup f \leq$ *ess* $\sup f$. But always *ess* $\sup f \leq \sup f$.

2. A metric space is *separable* if it contains a countable dense subset.

   Show that any separable Banach space is isometrically isomorphic to a closed subspace of ℓ^∞.

   Hint: Apply Theorem 5.8b to the elements of the countable dense subset.

   Solution: Let X be a separable Banach space and $\{x_n\}$ a countable dense subset.. By the Hahn-Banach Theorem (see Theorem 5.8b), for each n, there is a linear functional f_n s.t. $||f_n|| = 1$ and $f_n(x_n) = ||x_n||$.

   Let
   $$\Psi(x) = (f_1(x), f_2(x), ...)$$
   Ψ is linear since each f_i is linear.

   It suffices to show that Ψ is a BLT with $\|\Psi(x)\| = \|x\|$ for all $x \in X$.

   $||\Psi|| \leq 1$:
   $$||\Psi(x)|| = \sup_n ||f_n(x)|| \leq \sup_n ||f_n|| \; ||x|| = ||x||$$
   So Ψ is a BLT with $\|\Psi\| \leq 1$.

   Ψ preserves the norm:

   Fix $x \in X$. Let $\{x_{n_i}\} \subset \{x_n\}$ be a sequence converging to x. For each i we have

$\|x_{n_i}\| \geq \|\Psi(x_{n_i})\| \geq \|f_{n_i}(x_{n_i})\| = \|x_{n_i}\|$ so $\|\Psi(x_{n_i})\| = \|x_{n_i}\|$. We have shown that Ψ is bounded, hence continuous, and the norm is continuous, therefore

$$\|x\| \geq \|\Psi(x)\| = \lim_{i\to\infty} \|\Psi(x_{n_i})\| = \lim_{i\to\infty} \|x_{n_i}\| = \|x\|,$$

which yields the desired norm-preserving structure.

Ψ is injective since it preserves the norm, hence it preserves nonzero elements. So, Ψ is a linear norm-preserving map and thus is a bijection onto its image which is contained in ℓ^∞.

Since isometric isomorphisms preserve completeness, the image is complete and therefore closed.

3. Let X be a set and let T be the set of all topologies on X, partially ordered by inclusion. Recall that an element a of a partially ordered set (poset) is *maximal* if $a \leq b$ implies $a = b$. Similarly, a is *minimal* if $b \leq a$ implies $a = b$.

   (a) Show that T has a unique maximal element and a unique minimal element. Identify these elements. $\mathcal{T} \in T$ s.t. $\mathcal{T} \leq \mathcal{T}_1$ and unique maximal element. Identify this element in terms of $\mathcal{T}_1$ and $\mathcal{T}_2$.

   (b) Given any collection $\mathcal{E}$ of subsets of X, let $T_{\mathcal{E}}$ denote the set of all elements of T that contain $\mathcal{E}$. Order $T_{\mathcal{E}}$ by inclusion. Show that $T_{\mathcal{E}}$ has a unique minimal element. Identify this element in terms of $\mathcal{E}$.

   (c) Let $R \subset T$. Let T_R denote the collection of all topologies $\mathcal{T} \in T$ s.t. $\mathcal{T} \subset \mathcal{R}$ for all $\mathcal{R} \in R$. Order T_R by inclusion. Show that T_R has a unique maximal element. Identify this element in terms of R.

   (d) If $R = T_{\mathcal{E}}$ are the answers to part b and c the same?

   Solution:

   a. The discrete topology $\mathcal{T} = P(X)$, the set of all subsets, is maximal because clearly if $\mathcal{T} \leq \mathcal{S}$, then $\mathcal{S} = \mathcal{T}$. Similarly, the trivial topology $\mathcal{T} = \{\emptyset, X\}$ is a minimal element. Since $P(X) \geq \mathcal{S}$ for any topology $\mathcal{S}$, it is the unique maximal element. Similarly, the trivial topology $\mathcal{T} = \{\emptyset, X\}$ is the unique minimal element.

   For parts b and c, first note that the intersection of any collection of topologies is a topology.

   b. Let $\mathcal{T}_{\min}$ be the intersection of all elements of $T_{\mathcal{E}}$. $\mathcal{T}_{\min}$ is a topology and belongs to $T_{\mathcal{E}}$. If $\mathcal{T} \in T_{\mathcal{E}}$, then $\mathcal{T}_{\min} \subset \mathcal{T}$. So, if $\mathcal{T} \subset \mathcal{T}_{\min}$, then $\mathcal{T} = \mathcal{T}_{\min}$. So, $\mathcal{T}_{\min}$ is a minimal element.

   And if $\mathcal{T}$ is a minimal element of $T_{\mathcal{E}}$, then since $\mathcal{T}_{\min} \subset \mathcal{T}$, we have $\mathcal{T} = \mathcal{T}_{\min}$, and so $\mathcal{T}_{\min}$ is the unique minimal element.

   c. Let $\mathcal{T}_{\max}$ be the intersection of all elements of R. Then $\mathcal{T}_{\max}$ is a topology and belongs to T_R. If $\mathcal{T} \in T_R$, then $\mathcal{T} \subset \mathcal{T}_{\max}$. So, if $\mathcal{T}_{\max} \subset \mathcal{T}$, then $\mathcal{T} = \mathcal{T}_{\max}$, and so $\mathcal{T}$ is a maximal element.

If $\mathcal{T} \in T_R$ is maximal, then since $\mathcal{T} \subset \mathcal{T}_{\max}$, $\mathcal{T} = \mathcal{T}_{\max}$. So, $\mathcal{R}_{\max}$ is the unique maximal element of R.

d. Yes. They are both the intersection of all elements of $T_{\mathcal{E}}$.

4. Defn: For a vector space X, a *convex combination* of $x, y \in X$ is a point of the form $tx + (1-t)y$ such that $t \in [0,1]$. A subset S of a NVS X is *convex* if whenever $x, y \in S$ then every convex combination of x, y is in S.

   Let C be a convex set in a real normed vector space X and assume that $0 \in$ interior(C), i.e., for some $\epsilon > 0$, $B_\epsilon(0) \subset C$. For $x \in X$, let

$$p(x) = \inf_{x \in \lambda C} \lambda > 0$$

   (a) Show that $p(x)$ is a sublinear functional.

   (b) Give an example of a proper convex set C in $\mathbb{R}^2$ s.t. $p(x)$ is a semi-norm but not a norm.

   (c) Give an example of a proper convex set C in $\mathbb{R}^2$ s.t. $p(x)$ is a semilinear functional but not a semi-norm.

Solution:

a. $p(x)$ is called the *Minkowski functional.*

First observe that for all x, $p(x)$ is finite: if $x = 0$, then $x \in \lambda C$ for all $\lambda > 0$ and so $p(0) = 0$; if $x \neq 0$, then, since for some $\epsilon > 0$, $B_\epsilon(0) \subset C$, we have $(\epsilon/2)x/||x|| \in C$ and so $p(x) \leq 2||x||/\epsilon$.

Next observe that for all $\lambda > p(x)$, $x \in \lambda C$: there exists λ' s.t. $\lambda > \lambda' \geq p(x)$ and $x \in \lambda' C$; then $x = \lambda' c$ for some $c \in C$; then $x = \lambda(\lambda'/\lambda)c$ and $(\lambda'/\lambda)c \in C$ since it is a convex combination of c and 0: $(\lambda'/\lambda)c = (\lambda'/\lambda)c + (1 - (\lambda'/\lambda))0$; so $x \in \lambda C$.

positive homogeneity: Let $\mu > 0$. Then $x \in \lambda C$ iff $\mu x \in \mu\lambda C$ and so $\lambda > p(x)$ iff $\mu\lambda > p(\mu x)$, and so $p(\mu x) = \mu p(x)$.

subadditivity: Let $x, y \in X$. Given $\epsilon > 0$, choose $p(x) < \lambda < p(x) + \epsilon$, $p(y) < \mu < p(y) + \epsilon$. Then $x/\lambda, y/\mu \in C$ and so by convexity

$$x + y = (\lambda + \mu)(\frac{\lambda}{\lambda + \mu}(x/\lambda) + \frac{\mu}{\lambda + \mu}(x/\mu)) \in (\lambda + \mu)C$$

Thus,

$$p(x + y) \leq \lambda + \mu \leq p(x) + p(y) + 2\epsilon$$

Since this holds for all $\epsilon > 0$, we have $p(x + y) \leq p(x) + p(y)$.

For b and c, note that $p(x)$ is a semi-norm iff it is symmetric around the origin, i.e., $x \in C$ iff $-x \in C$; for a sublinear functional is a semi-norm iff for all x, $p(-x) = p(x)$.

b. Let C be the horizontal strip $\mathbb{R} \times [-1, 1]$. Then for all x $p(x, 0) = 0$. So, p is not a norm. But it is a semi-norm because it is symmetric around the origin.

c. Let C be a convex set which contains a nbhd. of the origin and is not symmetric around the origin. Specific example: the open unit disk shifted up by 1/2. Then $p((0, 3/4) = 1/2$, but $p(0, -3/4) = 3/2$ and so $p(-(0, 3/4)) \neq p(0, 3/4)$.

5. Show that $\mathbb{R}^n$ and $\mathbb{C}^n$, with each endowed by the Euclidean metric, are reflexive Banach spaces

   Solution: It suffices to show that $(\mathbb{R}^n)^* = \mathbb{R}^n$ and $(\mathbb{C}^n)^* = \mathbb{C}^n$.

   We do the case of $\mathbb{R}^n$; $\mathbb{C}^n$ is similar.

   For $y \in \mathbb{R}^n$, let L_y be the linear functional on $\mathbb{R}^n$ defined by $L_y(x) = x \cdot y$. Then by finite-dimensional Cauchy Schwartz, L_y is a BLF with norm at most $||y||$; but in fact, the norm is exactly $||y||$ because

$$L_y(y) = \sum_{i=1}^{n} (y_i)^2 = (||y||)^2$$

   Define:

$$\Psi : \mathbb{R}^n \to (\mathbb{R}^n)^*, y \mapsto L_y$$

   Then Ψ is linear:

$$L_{ay+bz}(x) = x \cdot (ay + bz) = ax \cdot y + bx \cdot z = aL_y(x) + bL_z(x)$$

   And norm-preserving, therefore injective, since $||L_y|| = ||y||$.

   Suffices to show that Ψ is surjective. Let $\phi \in (\mathbb{R}^n)^*$.

   Find $y \in \mathbb{R}^n$ s.t. $L_y = \phi$.

   Let $y_i = \phi(e^i)$. Then

$$\phi(x) = \sum_i x_i \phi(e_i) = \sum_i x_i y_i = L_y(x).$$

   Alternative solution: appeal to the result proven in class that $(L^p)^* = L^q$.

6. Show that $c_0^* = \ell^1$, more precisely that there is an isometric isomorphism from ℓ^1 onto c_0^* (here, c_0 has the sup norm and ℓ^1 has the ℓ^1 norm).

   Solution: For $\overline{a} \in \ell^1$ and $\overline{x} \in c_0$, define

$$\phi_{\overline{a}}(\overline{x}) = \sum_i a_i x_i,$$

   a convergent series. And

$$|\phi_{\overline{a}}(\overline{x})| \leq \sum_i |a_i| \ |x_i| \leq (\sum_i |a_i|)(\sup_i |x_i|) = |||\overline{a}||_{\ell_1} ||\overline{x}||$$

   Since $\phi_{\overline{a}}$ is clearly linear, $\phi_{\overline{a}} \in c_0^*$ and $||\phi|| \leq ||\overline{a}||_{\ell_1}$.

   On the other hand, letting $\overline{e}^n$ be the n-th canonical basis vector and

$\overline{x}^n = \sum_{i=1}^n sgn(a_i)\overline{e}^i$, which is in the unit sphere of c_0, we get

$$|\phi_{\overline{a}}(\overline{x}^n)| = \sum_{i=1}^n |a_i|$$

and so for all n,

$$\sup_{\overline{x}:||\overline{x}||=1} |\phi_{\overline{a}}(\overline{x})| \geq \sum_{i=1}^n |a_i|$$

and so $||\phi|| = ||\overline{a}||_{\ell_1}$.

Thus, $\overline{a} \mapsto \phi_{\overline{a}}$ is a linear norm-preserving, and therefore injective, mapping from ℓ^1 into c_0^*.

It remains to show that $\overline{a} \mapsto \phi_{\overline{a}}$ is surjective.

So let $\phi \in c_0^*$. and $a_n = \phi(\overline{e}^n)$. Let

$$\overline{a} = (a_1, a_2, \ldots)$$

We claim that $\overline{a} \in \ell^1$ and $\phi_{\overline{a}} = \phi$ (surjectivity then follows). Then by linearity for all n,

$$\sum_{i=1}^n |a_i| = \phi(\overline{x}^n) \leq ||\phi|| \; ||\overline{x}^n|| = ||\phi||$$

Thus, $\overline{a} \in \ell^1$.

Clearly ϕ and $\phi_{\overline{a}}$ agree on c_c, which is dense in c_0. Since both ϕ and $\phi_{\overline{a}}$ are continuous on c_0 they must agree on c_0.

7. (a) Let X be an NVS. Suppose that Y_1 and Y_2 are Banach spaces and there are isometric isomorphisms Φ_1 and Φ_2 from X into Y_1 and Y_2 whose images are dense in Y_1 and Y_2.

   Show that Y_1 and Y_2 are isometrically isomorphic.

   (b) Why is the closure of the image of the canonical embedding $X \to X^{**}$, $x \mapsto \hat{x}$ isometrically isomorphic to $(\overline{X}, ||\cdot||_{\overline{X}})$ (from problem 7 in HW1 and HW2)?

Solution:

a. By HW2 #7, there is an isometric isomorphism from X to a dense subspace of $\overline{X}$. By transitivity of isometric isomorphism it suffices to show that if there is an isometric isomorphism of X to a dense subspace of a Banach space Z, then Z is isometrically isomorphic to $\overline{X}$.

We may assume that X is itself a dense subpsace of Z. For $z \in Z$ let $\{x_n(z)\}$ be a sequence which converges to z. Then $x_n(z)$ is Cauchy. Thus, $[\{x_n(z)\}] \in \overline{X}$. Define

$$\Psi : Z \to \overline{X}$$

by

$$\Psi(z) = ([\{x_n(z)\}])$$

We claim that Ψ is an isometric isomorphism of Z onto $\overline{X}$.

Verify the following:

Well-defined: If x'_n is another sequence which converges to z, then $\lim_{n\to\infty} x_n(z)-x'_n = 0$ and so $\lim_{n\to\infty} ||x_n(z) - x'_n|| = 0$ and so $[\{x'_n\}] = [\{x_n(z)\}]$.

Linear: Let $z, z' \in Z$. Then $\lim_{n\to\infty} x_n(z) + x_n(z') = z + z'$ and so $\lim_{n\to\infty}(x_n(z) + x_n(z')) - x_n(z + z') = 0$ and thus

$$[\{x_n(z)] + [\{x_n(z')\}] = [\{x_n(z) + x_n(z')\}] = [\{x_n(z + z')\}].$$

Norm-preserving (and thus injective):

$$||[\{x_n(z)\}]|| = \lim_{n\to\infty} ||x_n(z)|| = ||z||$$

Surjective: Let $[\{x_n\}] \in \overline{X}$. Then $\{x_n\}$ is Cauchy in X and therefore converges to some $z \in Z$ since Z is complete. Then $[\{x_n\}] = [\{x_n(z)\}]$ and so

$$\Psi(z) = [\{x_n(z)\}] = [\{x_n\}]$$

b. The image of X via the canonical embedding is dense in its closure Z which is a Banach subspace of X^{**}. So, Z must be isometrically isomorphic to $\overline{X}$.

8. Defn: An *extreme point* of a convex set S is a point $z \in S$ such that z *cannot* be expressed as a nontrivial convex combination of distinct poins of S; here, non-trivial means that the λ in the convex combination is in $(0, 1)$

   (a) Show that a vector space isomorphism $T : X \to Y$ preserves the following:
      i. convex sets i.e., A is convex iff $T(A)$ is convex
      ii. extreme points of a convex set, i.e., if $E(A)$ denotes the set of extreme points of A, then $T(E(A)) = E(T(A))$
   (b) Show that if T is an isometric isomorphism, then it preserves the closed unit ball, i.e., the T-image of the closed unit ball of X is the closed unit ball of Y.
   (c) Show that the closed unit ball in an NVS is convex.
   (d) Find the extreme points of the closed unit balls in c_0, the Banash space of sequences that converge to 0, and in c, the Banach space of all convergent sequences.
   (e) Are c and c_0 isometrically isomorphic? Why or why not?

Solution:

a.

i. Since T is invertible, it suffices to show that if $T(A)$ is convex then A is convex. This follows from linearity:

Let $x, y \in A$ and $\lambda \in [0, 1]$ and

$$u = \lambda x + (1 - \lambda)y$$

Since T is linear,

$$T(u) = \lambda T(x) + (1-\lambda)T(y)$$

Since $T(x), T(y) \in T(A)$ which is convex, $T(u) \in T(A)$. Since T is invertible and T^{-1} is linear, $u \in A$. So A is convex.

ii. This follows from linearity and defn of extreme points.

Let v be an extreme point of $T(A)$ and $u = T^{-1}(v) \in A$.

If u were not an extreme point of A, then there would exist distinct $x, y \in A$ and $\lambda \in (0,1)$ s.t. $u = \lambda x + (1-\lambda)y$. By linearity of T,

$$v = T(u) = \lambda T(x) + (1-\lambda)T(y)$$

which is a non-trivial linear combination of distinct points in $T(A)$, a contradiction.

So, $E(T(A)) \subseteq T(E(A))$. But then replacing T by T^{-1} and A by $T(A)$ we get

$$E(A) = E(T^{-1}(T(A))) \subseteq T^{-1}(E(T(A))$$

. Apply T to this inclusion, we get $T(E(A)) \subseteq E(T(A))$.

b.. This follows immediately from the fact that an isometric isomorphism is norm-preserving.

c. Let x and y be in the closed unit ball. Then for any convex combination we have:

$$||\lambda x + (1-\lambda y|| \leq ||\lambda x|| + ||(1-\lambda y|| = \lambda||||x|| + (1-\lambda)||y|| \leq 1.$$

d. Let e_n denote the n-th canonical basis vector in $\mathbb{R}^\infty$:

$$e_n = (0,0,0,\ldots,0,0,0,1,0,0,0,\ldots).$$

Suppose that x belongs to the closed unit ball of c and for some n, $|x_n| < 1$, then for sufficiently small $\epsilon > 0$, both $x_{\pm,\epsilon} = x \pm \epsilon e_n$ belong to the unit ball of c. Moreover, x is the average of these points:

$$x = (1/2)(x_{+,\epsilon} + (x_{-,\epsilon}));$$

thus such an x is not an extreme point of the unit ball in c.

For the exact same reason, if x belongs to the closed unit ball of c_0 and for some n, $|x_n| < 1$, then such an x is not an extreme point of the unit ball in c_0.

It follows that the only extreme points of the closed unit balls of either c_0 or c are those x s.t. for all n, $|x_n| = 1$. Thus, the set of extreme points of the closed unit ball of c_0 is empty.

For the space c, each of these sequences is extreme because if such an x is a non-trivial convex combination of two distinct sequences, then for at least one of those sequences, call it y, and some n, $|y_n| > 1$ and thus is not in the unit ball. So, the set of extreme points of the closed unit ball of c is the set of sequences x s.t. for all n $|x_n| = 1$ and for all sufficiently large n, the sequence is 1 or for all sufficiently large n, the sequence is -1.

e. No. By Parts a, b and c, an isometric isomorphism would map the extreme points of the unit ball of c to the extreme points of c_0. But, by part d, the former is non-empty and the latter is empty.