Math 421/510 Homework 3: Due on Friday, Feb. 15, 11AM

1. Let μ be Lebesgue measure on $[0,1]$.

   (a) Show that if $f, g \in C([0,1])$ and $f = g$ μ-a.e., then $f = g$, and thus for any $1 \le p \le \infty$, $C([0,1])$ may be regarded as a subspace of $L^p([0,1],\mu)$.

   (b) Show that the sup norm and L^1 norm on $C([0,1])$ are not equivalent.

   (c) Are the sup norm and L^∞ norm on $C([0,1])$ equivalent?

2. A metric space is *separable* if it contains a countable dense subset.

   Show that any separable Banach space is isometrically isomorphic to a closed subspace of ℓ^∞.

   Hint: Apply Theorem 5.8b to the elements of the countable dense subset.

3. Let X be a set and let T be the set of all topologies on X, partially ordered by inclusion. Recall that an element a of a partially ordered set (poset) is *maximal* if $a \le b$ implies $a = b$. Similarly, a is *minimal* if $b \le a$ implies $a = b$.

   (a) Show that T has a unique maximal element and a unique minimal element. Identify these elements.

   (b) Given any collection $\mathcal{E}$ of subsets of X, let $T_\mathcal{E}$ denote the set of all elements of T that contain $\mathcal{E}$. Order $T_\mathcal{E}$ by inclusion. Show that $T_\mathcal{E}$ has a unique minimal element. Identify this element in terms of $\mathcal{E}$.

   (c) Let $R \subset T$. Let F_R denote the collection of all topologies $\mathcal{T} \in T$ s.t. $\mathcal{T} \subset \mathcal{R}$ for all $\mathcal{R} \in R$. Order F_R by inclusion. Show that F_R has a unique maximal element. Identify this element in terms of R.

   (d) If $R = T_\mathcal{E}$ are the answers to part b and c the same?

4. Defn: For a vector space X, a *convex combination* of $x, y \in X$ is a point of the form $tx + (1-t)y$ such that $t \in [0,1]$. A subset S of X is *convex* if whenever $x, y \in S$ then every convex combination of x, y is in S.

   Let C be a convex set in a real NVS X and assume that $0 \in$ interior(C), i.e., for some $\epsilon > 0$, $B_\epsilon(0) \subset C$. For $x \in X$, let

   $$p(x) = \inf_{x \in \lambda C} \lambda > 0$$

   (a) Show that $p(x)$ is a sublinear functional.

   (b) Give an example of a proper convex subset C in $\mathbb{R}^2$ s.t. $p(x)$ is a semi-norm but not a norm.

   (c) Give an example of a proper convex subset C in $\mathbb{R}^2$ s.t. $p(x)$ is a sublinear functional but not a semi-norm.

5. Show that $\mathbb{R}^n$ and $\mathbb{C}^n$, with each endowed by the Euclidean metric, are reflexive Banach spaces

6. Show that $c_0^* = \ell^1$, more precisely that there is an isometric isomorphism from ℓ^1 onto c_0^* (here, c_0 has the sup norm and ℓ^1 has the ℓ^1 norm).

7. (a) Let X be an NVS. Suppose that Y_1 and Y_2 are Banach spaces and there are isometric isomorphisms Φ_1 and Φ_2 from X into Y_1 and Y_2 whose images are dense in Y_1 and Y_2.

   Show that Y_1 and Y_2 are isometrically isomorphic.

   (b) Why is the closure of the image of the canonical embedding $X \to X^{**}$, $x \mapsto \hat{x}$ isometrically isomorphic to $(\overline{X}, || \cdot ||_{\overline{X}})$ (from problem 7 in HW1 and HW2)?

8. Defn: An *extreme point* of a convex set S is a point $z \in S$ such that z *cannot* be expressed as a nontrivial convex combination of distinct poins of S; here, non-trivial means that the λ in the convex combination is in $(0, 1)$

   (a) Show that a vector space isomorphism $T : X \to Y$ preserves the following:

      i. convex sets i.e., A is convex iff $T(A)$ is convex

      ii. extreme points of a convex set, i.e., if $E(A)$ denotes the set of extreme points of A, then $T(E(A)) = E(T(A))$

   (b) Show that if T is an isometric isomorphism, then it preserves the closed unit ball, i.e., the T-image of the closed unit ball of X is the closed unit ball of Y.

   (c) Show that the closed unit ball in an NVS is convex.

   (d) Find the extreme points of the closed unit balls in c_0, the Banash space of sequences that converge to 0, and in c, the Banach space of all convergent sequences.

   (e) Are c and c_0 isometrically isomorphic? Why or why not?