Math 421/510 Homework 3: Due on Friday, Feb. 15, 11AM

1. Let $\mu$ be Lebesgue measure on $[0, 1]$.
   (a) Show that if $f, g \in C([0, 1])$ and $f = g \mu$-a.e., then $f = g$, and thus for any $1 \leq p \leq \infty$, $C([0, 1])$ may be regarded as a subspace of $L^p([0, 1], \mu)$.
   (b) Show that the sup norm and $L^1$ norm on $C([0, 1])$ are not equivalent.
   (c) Are the sup norm and $L^\infty$ norm on $C([0, 1])$ equivalent?

2. A metric space is separable if it contains a countable dense subset.
   Show that any separable Banach space is isometrically isomorphic to a closed subspace of $\ell^\infty$.
   Hint: Apply Theorem 5.8b to the elements of the countable dense subset.

3. Let $X$ be a set and let $T$ be the set of all topologies on $X$, partially ordered by inclusion.
   Recall that an element $a$ of a partially ordered set (poset) is maximal if $a \leq b$ implies $a = b$. Similarly, $a$ is minimal if $b \leq a$ implies $a = b$.
   (a) Show that $T$ has a unique maximal element and a unique minimal element. Identify these elements.
   (b) Given any collection $\mathcal{E}$ of subsets of $X$, let $T_\mathcal{E}$ denote the set of all elements of $T$ that contain $\mathcal{E}$. Order $T_\mathcal{E}$ by inclusion. Show that $T_\mathcal{E}$ has a unique minimal element. Identify this element in terms of $\mathcal{E}$.
   (c) Let $R \subset T$. Let $F_R$ denote the collection of all topologies $\mathcal{T} \in T$ s.t. $\mathcal{T} \subset \mathcal{R}$ for all $\mathcal{R} \in R$. Order $F_R$ by inclusion. Show that $F_R$ has a unique maximal element. Identify this element in terms of $R$.
   (d) If $R = T_\mathcal{E}$ are the answers to part b and c the same?

4. Defn: For a vector space $X$, a convex combination of $x, y \in X$ is a point of the form $tx + (1 - t)y$ such that $t \in [0, 1]$. A subset $S$ of $X$ is convex if whenever $x, y \in S$ then every convex combination of $x, y$ is in $S$.
   Let $C$ be a convex set in a real NVS $X$ and assume that $0 \in \text{interior}(C)$, i.e., for some $\epsilon > 0$, $B_\epsilon(0) \subset C$. For $x \in X$, let
   $$p(x) = \inf_{x \in \lambda C} \lambda > 0$$
   (a) Show that $p(x)$ is a sublinear functional.
   (b) Give an example of a proper convex set $C$ in $\mathbb{R}^2$ s.t. $p(x)$ is a semi-norm but not a norm.
   (c) Give an example of a proper convex set $C$ in $\mathbb{R}^2$ s.t. $p(x)$ is a semilinear functional but not a semi-norm.
5. Show that \( \mathbb{R}^n \) and \( \mathbb{C}^n \), with each endowed by the Euclidean metric, are reflexive Banach spaces.

6. Show that \( c_0^* = \ell^1 \), more precisely that there is an isometric isomorphism from \( \ell^1 \) onto \( c_0^* \) (here, \( c_0 \) has the sup norm and \( \ell^1 \) has the \( \ell^1 \) norm).

7. (a) Let \( X \) be an NVS. Suppose that \( Y_1 \) and \( Y_2 \) are Banach spaces and there are isometric isomorphisms \( \Phi_1 \) and \( \Phi_2 \) from \( X \) into \( Y_1 \) and \( Y_2 \) whose images are dense in \( Y_1 \) and \( Y_2 \).

Show that \( Y_1 \) and \( Y_2 \) are isometrically isomorphic.

(b) Why is the closure of the image of the canonical embedding \( X \to X^{**}, x \mapsto \hat{x} \) isometrically isomorphic to \( (\overline{X}, \| \cdot \|_{\overline{X}}) \) (from problem 7 in HW1 and HW2)?

8. Defn: An extreme point of a convex set \( S \) is a point \( z \in S \) such that \( z \) cannot be expressed as a nontrivial convex combination of distinct points of \( S \); here, non-trivial means that the \( \lambda \) in the convex combination is in \((0, 1)\)

(a) Show that a vector space isomorphism \( T : X \to Y \) preserves the following:

i. convex sets i.e., \( A \) is convex iff \( T(A) \) is convex

ii. extreme points of a convex set, i.e., if \( E(A) \) denotes the set of extreme points of \( A \), then \( T(E(A)) = E(T(A)) \)

(b) Show that if \( T \) is an isometric isomorphism, then it preserves the closed unit ball, i.e., the \( T \)-image of the closed unit ball of \( X \) is the closed unit ball of \( Y \).

(c) Show that the closed unit ball in an NVS is convex.

(d) Find the extreme points of the closed unit balls in \( c_0 \), the Banach space of sequences that converge to 0, and in \( c \), the Banach space of all convergent sequences.

(e) Are \( c \) and \( c_0 \) isometrically isomorphic? Why or why not?