Math 421/510, Homework 2, due Friday, Feb. 1, 11AM

1. Let $L^\infty = L^\infty((X,\mu);\mathbb{R})$, for some measure space $(X,\mu)$, with the essential sup norm $||\cdot||_\infty$. For $g \in L^\infty$, define $T : L^\infty \to L^\infty$ by $T(f) = fg$. Show that $T$ is well-defined, a BLT and $||T|| = ||g||_\infty$

   Solution: First note that $T$ is well-defined, i.e., if $f_1 = f_2$ a.e. and $g_1 = g_2$ a.e., then $f_1 g_1 = f_2 g_2$ a.e.

   Clearly $T$ is linear.

   We may assume that $\sup |g| = ||g||_\infty$. For any $f \in L^\infty$, we may assume that $\sup |f| = ||f||_\infty$. Then

$$||Tf||_\infty = ||fg||_\infty \le \sup |fg| \le (\sup |f|)(\sup |g|) = ||f||_\infty \; ||g||_\infty$$

   Thus $||T|| \le ||g||_\infty$. But for $f = 1$, the constant function,

$$||Tf||_\infty = ||g||_\infty = ||g||_\infty \; ||f||_\infty$$

   So, $T$ is a BLT and $||T|| = ||g||_\infty$

2. (a) Show that if $T$ is a BLT from an NVS $(X, ||\cdot||_1)$ to $(Y||\cdot||_2)$, $||\cdot||_1$ is equivalent to $||\cdot||_3$ and $||\cdot||_2$ is equivalent to $||\cdot||_4$, then $T$ is a BLT from $(X, ||\cdot||_3)$ to $(X, ||\cdot||_4)$.

   (b) Show that any linear transformation from a finite-dimensional NVS to any NVS is a BLT.

   Solution:

   a. Let $||T||_{12}$ denote the norm of $T$ with respect norms 1 and 2, and $||T||_{34}$ with respect to norms 3 and 4.

   For some $C, K > 0$,

$$||Tx||_4 \le C||Tx||_2 \le C||T||_{12} \; ||x||_1 \le CK||T||_{12} \; ||x||_3$$

   Thus $T$ is a BLT wrt norms 3 and 4 with $||T_{34}|| \le CK|T||_{12}$.

   b. By part a, we may assume that the finite-dimensional NVS is $\mathbb{R}^n$ with the $\ell^1$ norm. Any $x \in X$ may be expressed $x = \sum_i a_i e_i$. Let $M = \max ||T(e_i)||$. Then,

$$||T(x)|| \le \sum_i |a_i| \; ||T(e_i)|| \le M||x||$$

   and so $T$ is a BLT with $||T|| \le M$.

3. Let $T : X \to Y$ be a linear transformation from one vector space $X$ to another $Y$. Assume that each of $X$ and $Y$ are endowed with norms and so may be considered an NVS. Show that the following are equivalent.

   (a) $T$ is a BLT

(b) The image of every bounded set is bounded.

(c) For some open set $U$, containing 0, in $X$, $T(U)$ is bounded in $Y$.

Solution:

a implies b: Any bounded set $B$ in $X$ is contained in a closed ball, of some radius $K$, centered at 0. Since for all $x \in B$, $||Tx|| \leq ||T||\ ||x||$, we have $||Tx|| \leq K||T||$. Thus the image of $B$ is contained in the closed ball of radius $||T||K$, a bounded set.

b implies c: The open unit ball $U$ contains 0 and is clearly bounded. Thus, given b, its image is bounded.

c implies a: Since any open set is a union of open balls, $U$ contains an open ball $B$ containing 0. But then $B$ must contain a closed ball, $\overline{B}_\delta(0)$, centered at 0. Since $T(U)$ is bounded, $T(\overline{B}_\delta(0))$ is bounded and thus must be contained in $\overline{B}_K(0)$ for some $K$.

So, if $x \in X$ and $x \neq 0$, then $||T(\delta x/||x||)|| \leq K$ and so $||T(x)|| \leq (K/\delta)||x||$ and so $T$ is a BLT with $||T|| \leq K/\delta$.

4. A *homeomorphic isomorphism* from one NVS $X$ to another $Y$ is a vector space isomorphism that is also a homeomorphism. We say that $X$ and $Y$ are *homeomorphically isomorphic* (which is sometimes just called an isomorphism in functional analysis) if there is a homeomorphic isomorphism from $X$ to $Y$.

   (a) Show that homeomorphic isomorphism is an equivalence relation.

   (b) Show that a vector space isomorphism is a homeomorphic isomorphism iff it and its inverse are BLTs.

   (c) Show that if $X$ and $Y$ are homeomorphically isomorphic, then $X$ is a Banach space iff $Y$ is.

   (d) Let $X = Y = c_c$, the set of sequences that are eventually 0, considered as NVS with the sup norm. Let $T : X \to Y$ be defined by

   $$T(\{x_1, x_2, x_3, \dots,\}) = \{x_1, x_2/2, x_3/3, \dots\}$$

   Show that $T$ is linear, bijective, continuous, and has a linear inverse but that $T$ is *not* a homeomorphic isomorphism.

   (later on, we will see that the open mapping theorem implies that this can't happen when $X$ and $Y$ are Banach spaces).

Solution:

a.

i. Reflexivity: the identity is both a vector space isomorphism and a homeomorphism.

ii. Symmetry: the inverse of a vector isomorphism is a vector isomorphism and the inverse of a homeomorphism is a homeomorphism.

iii. Transitivity: the composition of vector space isomorphisms is a vector space isomorphism and the composition of homeomorphisms is a homeomorphism.

b. Continuity of a linear transformation is equivalent to BLT.

c. If $T$ is a BLT, then for all $x$, $||Tx|| \le ||T||\ ||x||$ and so $T$ maps Cauchy sequences to Cauchy sequences and convergent sequences to convergent sequences. So, if $T$ is homeomorphic isomorphism, then $X$ is Banach iff $Y$ is.

d. $T$ is clearly linear. It is continuous because it is a BLT:

$$||Tx|| = \sup_{n:x_n \neq 0} |x_n/n| \le \sup_{n:x_n \neq 0} |x_n| = ||x||$$

and so $||T|| \le 1$ (in fact, $||T|| = 1$).

It has a linear inverse:

$$T^{-1}(\{y_1, y_2, y_3, \ldots\}) = \{y_1, 2y_2, 3y_3, \ldots\}$$

and in particular is biejctive.

But its inverse is not continuous because it is not a BLT:

Let $e_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, i.e., a 1 in position $n$ and 0's in the other positions.

Then $\frac{||T^{-1}(e_n)||}{||e_n||} = n$ and so $||T^{-1}|| = \infty$.

5. (a) Let $c$ denote the set of convergent sequences with the sup norm. Show that $c$ is a Banach space.

   (b) Show that $c$ and $c_0$ are homeomorphically isomorphic

Solution:

a. We claim that $c$ is a closed subspace of the Banach space $\ell^\infty$ and is thus a Banach space. It is clearly a subspace. To see that it is closed, let $x^n \in c$ converge to $x \in \ell^\infty$, i.e. $x^n = \{x^n_m\}$ converges to $x = \{x_m\}$ uniformly. Then for all $\epsilon > 0$, there exists $N$ s.t for all $n \ge N$ and all $m$

$$|x^n_m - x_m| < \epsilon$$

Since $x^N$ is convergent, it is Cauchy and so there exists $M$ s.t. for all $m_1, m_2 \ge M$

$$|x^N_{m_1} - x^N_{m_2}| < \epsilon$$

Thus, by the triangle inequality

$$|x_{m_1} - x_{m_2}| \le |x_{m_1} - x^N_{m_1}| + |x^N_{m_1} - x^N_{m_2}| + |x^N_{m_2} - x_{m_2}| < 3\epsilon$$

Thus $x$ is Cauchy and thus converges.

b. For each $x \in c$, let $\ell(x) = \lim_n x_n$. Define $\Phi : c \to c_0$ by

$$\Phi(x) = (\ell(x), x_1 - \ell(x), x_2 - \ell(x), \ldots, x_n - \ell(x), \ldots).$$

$\Phi(x) \in c_0$ because $\lim(x_n - \ell(x)) = 0$.

$\Phi$ is clearly linear. It is surjective because given $y \in c_0$, letting

$$x = (y_1, y_2 + y_1, y_3 + y_1, \ldots), \tag{1}$$

we have $\lim_n x_n = \lim_n y_n + y_1 = y_1$, and so $x \in c$, and

$$\Phi(x) = (y_1, x_2 - y_1, x_3 - y_1, \ldots) = (y_1, y_2, y_3, \ldots) = y.$$

It is injective because if $\Phi(x) = \Phi(x')$, then $L := \lim_n x_n = \lim_n x'_n$ and so each $x_n - L = x'_n - L$ and so each $x_n = x'_n$.

So, $\Phi$ is a bijective linear mapping and so it is a vector space isomorphism.

It is a BLT because

$$||\Phi(x)|| = \max(|\ell(x)|, \sup_n |\ell(x) - x_n|)$$
$$\leq \max(\sup_n |x_n|, 2\sup_n |x_n|) \leq 2\sup_n |x_n| = 2||x||$$

Finally, its inverse is a BLT:

$$\Phi^{-1}(y) = (y_1, y_2 + y_1, y_3 + y_1, \ldots)$$

and so

$$||\Phi^{-1}(y)|| = \max(|y_1|, \sup_n |y_{n+1} + y_1|) \leq |y_1| + \sup_n |y_n| \leq 2||y||$$

6. An *isometric isomorphism* from one NVS $X$ to another $Y$ is a bijective norm-preserving (i.e for all $x$, $||Tx|| = ||x||$) vector space isomorphism. If there exists such a map we say that $X$ and $Y$ are *isometrically isomorphic.*

   (a) Show that isometric isomorphism is an equivalence relation on normed vector spaces.

   (b) Show that a vector space isomorphism is an isometric isomorphism iff it and its inverse are BLTs with norm $= 1$.

   (c) Show that isometric isomorphism implies homeomorphic isomorphism.

   (d) Let $X$ be the space of analytic functions on the open unit disk $D$ which extend to continuous functions on the closed unit disk $\overline{D}$. Let $Y$ be the space of continuous functions on the unit circle which extend to continuous functions on $\overline{D}$ which are analytic on $D$ (both $X$ and $Y$ considered as NVS with the sup norm).

   Show that $X$ and $Y$ are Banach spaces and are isometrically isomorphic.

Solution:

a. Of course, vector space isomorphism is an equivalence relation.

Clearly the identity is norm preserving, the inverse of a norm preserving map is norm preserving and the composition of two norm preserving maps is norm preserving. So isometric isomorphism is an equivalence relation.

b. (If part) since $||T|| = 1$, for all $x$, $||Tx|| \leq ||x||$; since $||T^{-1}|| = 1$, for all $y$, $||T^{-1}y|| \leq ||y||$ and so replacing $T^{-1}y$ with $x$ and $y$ with $Tx$, we get for all $x$, $||x|| \leq ||Tx||$ and so $||Tx|| = ||x||$. Thus, $T$ is an isometric isomorphism.

(Only if part) Immediate.

c. This follows immediately from problems 4b and 6b.

d. Because uniform limits of analytic functions are analytic and uniform limits of continuous functions are continuous, $X$ is a closed subspace of the bounded complex-valued functions on $\overline{D}$ and thus is a Banach space.

Define $T : X \to Y$ by $Tf = f|_S$. The map $T$ is clearly linear and is bijective by definition. It is injective by the Cauchy integral formula from complex analysis: by continuity, any complex integral around the boundary, $S^1$ is uniformly approximable by a complex integral around the circle of radius $1 - \epsilon$. By the maximum modulus principle in complex analysis, $T$ is norm-preserving. Thus, $X$ and $Y$ are isometrically isomorphic and thus $Y$ is also a Banach space by problems 4c and 6c.

7. Continuation of problem 7 of HW1:

   d Show that $(\overline{X}, ||\cdot||_{\overline{X}})$ is complete.
   Hint: A Cauchy sequence in $\overline{X}$ is a sequence $x^m = [\{x^m_n\}] \in \overline{X}$ s.t.
   $$\lim_{m_1, m_2 \to \infty} \lim_{n \to \infty} ||x^{m_1}_n - x^{m_2}_n|| = 0 \text{ as } m_1, m_2 \to \infty.$$
   Find the limit of the Cauchy sequence by a diagonal argument: $y_m = x^m_{k_m}$ where $k_m$ is chosen appropriately.

   e Show that the map $h : X \to \overline{X}$, defined by $x \mapsto [(x, x, x, \ldots)]$ is an isometric isomorphism onto its image which is dense in $\overline{X}$.

   f Show that $h$ is onto iff $(X, ||\cdot||)$ is a Banach space.

Solution:

d. Let $[x^m]$ be a Cauchy sequence in $\overline{X}$. For fixed $m$, $x^m = \{x^m_n\}$ is a Cauchy sequence in $X$ and so there exists $k_m$ s.t. if $n_1, n_2 \geq k_m$, then
$$||x^m_{n_1} - x^m_{n_2}|| < 1/m.$$

We claim that the sequence $y = \{y_m\}$ is Cauchy in $X$. Let $\epsilon > 0$. Choose $M > 0$ s.t. $1/M < \epsilon$ and if $m_1, m_2 \geq M$, then
$$\lim_{j \to \infty} ||x^{m_1}_j - x^{m_2}_j|| < \epsilon$$

Fix $m_1, m_2 \geq M$, Then choose $J \geq k_{m_1}, k_{m_2}$ s.t. if $j \geq J$, then $||x^{m_1}_j - x^{m_2}_j|| < \epsilon$. Then
$$||y_{m_1} - y_{m_2}|| = ||x^{m_1}_{k_{m_1}} - x^{m_2}_{k_{m_2}}||$$
$$\leq ||x^{m_1}_{k_{m_1}} - x^{m_1}_J|| + ||x^{m_1}_J - x^{m_2}_J|| + ||x^{m_2}_J - x^{m_2}_{k_{m_2}}||$$

$$< 1/m_1 + \epsilon + 1/m_2 < 3\epsilon,$$

So, $y$ is Cauchy.

Finally, we claim that $[\{x^n\}]$ converges to $[y]$.

$$||[x^n] - [y]|| = \lim_{j\to\infty} ||x_j^n - y_j|| \leq \limsup_{j\to\infty} ||x_j^n - x_{k_n}^n|| + \limsup_{j\to\infty} ||y_n - y_j||$$

By defn. of $k_n$ the first term is at most $1/n$ and the second term is small for large $n$ since $y$ is Cauchy.

e. $h$ is clearly injective and thus bijective onto its image. It is clearly linear and so is a bijective linear transformation which is thus a vector space isomorphism onto its image. And it is norm preserving by the definition of the norm on equivalence classes of Cauchy sequences.

To show that its image is dense in $\overline{X}$, consider any given any equivalence class of Cauchy sequences $[\{x_n\}]$ and any $\epsilon > 0$. Choose $N$ s.t. if $m, n \geq N$, then $||x_m - x_n|| < \epsilon$. Let $y = \{x_N, x_N, x_N, \ldots\}$. Then $[y]$ is in the image of $h$ and in $\overline{X}$,

$$||[x] - [y]|| = \lim_{j\to\infty} ||x_j - x_N|| < \epsilon.$$

f. If $h$ is onto, then it is an isometric isomorphism onto $\overline{X}$ which is a Banach space and thus $X$ is also a Banach space.

Conversely, suppose that $X$ is a Banach space. Then every Cauchy sequence $\{x_n\}$ in $X$ converges to a point $x$ in $X$ and $\{x_n\}$ is equivalent to $\{x, x, \ldots\}$. Thus, $[\{x_n\}] = [(x, x, \ldots)]$ and thus is in the image of $h$. Thus, $h$ is onto.

8. Let $\Omega$ be a metric space. Recall that

$$C_0(\Omega) = \{\textit{continuous } f : \Omega \to R :, \ \forall \epsilon > 0 \ \exists \text{ compact } L \subset \Omega : \ \forall x \notin L \ |f(x)| < \epsilon\}$$

is a Banach space with the sup norm.

(a) Show that

$$C_0(\Omega) = \{\textit{continuous } f : \Omega \to \mathbb{R} : \ \forall \epsilon > 0, \ f^{-1}((-\epsilon, \epsilon)^c) \textit{ is compact}\}$$

(b) Let

$$K_0(\Omega) = \{\textit{bounded, continuous } f : \Omega \to \mathbb{R} : \lim_{x\to\infty} f(x) = 0\}$$

where $x \to \infty$ means that $d(x, p) \to \infty$ for any fixed $p \in \Omega$.

Show that $(K_0(\Omega), ||\cdot||_{\sup})$ is a Banach space.

(c) Show that $C_0(\Omega) \subset K_0(\Omega)$.

(d) Give an example to show that $K_0(\Omega)$ need not be contained in $C_0(\Omega)$.

Hint: Define a continuous function on an infinite dimensional NVS $\Omega$ which is constant on the unit ball and decays to 0 as $||x|| \to \infty$.

Solution:

a. Suppose that $f$ belongs to the RHS. Then given any $\epsilon > 0$, $L := f^{-1}((-\epsilon, \epsilon)^c)$ is compact and if $x \notin L$, then $|f(x)| < \epsilon$.

Conversely, let $f \in C_0(\Omega)$ and $L$ a compact set for $f$ and $\epsilon$. Then $f^{-1}((-\epsilon, \epsilon)^c) \subset L$ and so $f^{-1}((-\epsilon, \epsilon)^c)$ is a closed subset, due to the continuity of $f$, of a compact set and thus is compact.

b. Suffices to show that $K_0(\Omega)$ is a closed subspace of $BC(\Omega)$

It is clearly a subspace of $BC(\Omega)$.

Verify that $K_0(\Omega)$ is a closed subset of $BC(\Omega)$:

Let $f_n \in K_0(\Omega)$ s.t. $f_n$ converges uniformly to $f \in BC(\Omega)$. Let $\epsilon > 0$. Then for some $N$, $||f_N - f|| < \epsilon$.

Then there is some $M_{f_N,\epsilon}$ s.t. if $d(x,p) \geq M_{f_N,\epsilon}$, then $|f_N(x)| < \epsilon$.

Then for $d(x,p) \geq M_{f_N,\epsilon}$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq ||f - f_N|| + |f_N(x)| < 2\epsilon$$

So, $f \in K_0(\Omega)$, and so $K_0(\Omega)$ is a closed subspace of $BC(\Omega)$ and thus is a Banach space. $\square$

c. Let $f \in C_0(\Omega)$. Given $\epsilon > 0$, $f^{-1}((-\epsilon, \epsilon)^c)$ is compact and therefore bounded, i.e. $f^{-1}(-\epsilon, \epsilon)^c) \subset B_M(p)$ for some $M$. Thus $f \in K_0(\Omega)$.

d. Example:

Let $\Omega$ be any infinite dimensional Banach space. Let $f : \Omega \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{cc} 1 & ||x|| \leq 1 \\ 1/||x|| & ||x|| \geq 1 \end{array} \right\}$$

Then $f$ is continuous because it is continuous on the closed unit ball, on the complement of the open unit ball and agrees on the unit sphere, which is closed. It is clearly bounded and $\lim_{||x|| \to \infty} f(x) = 0$. So, $f \in K_0(\Omega)$.

For all $\epsilon > 1$, $f^{-1}((-\epsilon, \epsilon)^c) = \{||x|| \leq 1/\epsilon\}$ which is not compact. Thus, $f \notin C_0(\Omega)$.

9. (a) Let $X = Y = c_0$, the set of sequences that converge to 0, and let $T : X \to Y$ be defined by

$$T(\{x_n\}) = \{x_n(1 - 1/n)\}$$

Show that $T$ is a BLT with $||T|| = 1$, but for any $x$ in the unit sphere, $||Tx|| < 1$.

(b) Let $X$ be a reflexive Banach space and $Y = K$. Show that if $T : X \to Y$ is a BLF with $||T|| = 1$, then there exists $x$ in the unit sphere of $X$ s.t. $||Tx|| = 1$.

Solution: a. $||T|| = 1$:

$||T(x)|| = \sup |x_n(1 - 1/n)| \leq \sup |x_n|$ and so $||T|| \leq 1$.

For the $n$-th canonical basis vector $e_n$, $||T(e_n)|| = 1 - 1/n$ and so $||T|| \geq 1$.

Let $x$ be in the unit sphere of $c_0$. Given $0 < a < 1$, there exists $N$ s.t. for $n \geq N$, $|x_n| < a$. Then

$$||T(x)|| \leq \max(|x_1(0)|, |x_2(1/2)|, \ldots, |x_N(1 - 1/N)|, a)$$

$$\leq \max(1 - 1/N, a) < 1.$$

So, $||T(x)|| < 1 = ||T||$. □

b. Let $Y = X^*$. By Theorem 5.8b, applied to $Y$, for all $T \in Y$ with $||T|| = 1$, there exists $\phi \in Y^*$, s.t. $||\phi|| = 1$ and $\phi(T) = ||T|| = 1$. Since $X$ is reflexive, $Y^*$ can be identified with $X$ where $\phi$ corresponds to $\hat{x}$ for some $x \in X$, so that $||x|| = 1$ and $T(x) = \hat{x}(T) = ||T|| = 1$.