Math 421/510, Homework 2, due Friday, Feb. 1, 11AM

1. Let $L^\infty = L^\infty((X,\mu);\mathbb{R})$, for some measure space (X,μ), with the essential sup norm $||\cdot||_\infty$. For $g \in L^\infty$, define $T : L^\infty \to L^\infty$ by $T(f) = fg$. Show that T is well-defined, a BLT and $||T|| = ||g||_\infty$

2. (a) Show that if T is a BLT from an NVS $(X, ||\cdot||_1)$ to $(Y||\cdot||_2)$, $||\cdot||_1$ is equivalent to $||\cdot||_3$ and $||\cdot||_2$ is equivalent to $||\cdot||_4$, then T is a BLT from $(X, ||\cdot||_3)$ to $(X, ||\cdot||_4)$.

   (b) Show that any linear transformation from a finite-dimensional NVS to any NVS is a BLT.

3. Let $T : X \to Y$ be a linear transformation from one vector space X to another Y. Assume that each of X and Y are endowed with norms and so may be considered an NVS. Show that the following are equivalent.

   (a) T is a BLT

   (b) The image of every bounded set is bounded.

   (c) For some open set U, containing 0, in X, $T(U)$ is bounded in Y.

4. A *homeomorphic isomorphism* from one NVS X to another Y is a vector space isomorphism that is also a homeomorphism. We say that X and Y are *homeomorphically isomorphic* (which is sometimes just called an isomorphism in functional analysis) if there is a homeomorphic isomorphism from X to Y.

   (a) Show that homeomorphic isomorphism is an equivalence relation on normed vector spaces.

   (b) Show that a vector space isomorphism is a homeomorphic isomorphism iff it and its inverse are BLTs.

   (c) Show that if X and Y are homeomorphically isomorphic, then X is a Banach space iff Y is.

   (d) Let $X = Y = c_c$, the set of sequences that are eventually 0, considered as NVS with the sup norm. Let $T : X \to Y$ be defined by

   $$T(\{x_1, x_2, x_3, \ldots,\}) = \{x_1, x_2/2, x_3/3, \ldots\}$$

   Show that T is linear, bijective, continuous, and has a linear inverse but that T is *not* a homeomorphic isomorphism.

   (later on, we will see that the open mapping theorem implies that this can't happen when X and Y are Banach spaces).

5. (a) Let c denote the set of convergent sequences with the sup norm. Show that c is a Banach space.

   (b) Show that c and c_0 are homeomorphically isomorphic

6. An *isometric isomorphism* from one NVS X to another Y is a bijective norm-preserving (i.e for all x, $||Tx|| = ||x||$) vector space isomorphism. If there exists such a map we say that X and Y are *isometrically isomorphic.*

   (a) Show that isometric isomorphism is an equivalence relation on normed vector spaces.

   (b) Show that a vector space isomorphism is an isometric isomorphism iff it and its inverse are BLTs with norm = 1.

   (c) Show that isometric isomorphism implies homeomorphic isomorphism.

   (d) Let X be the space of analytic functions on the open unit disk D which extend to continuous functions on the closed unit disk $\overline{D}$. Let Y be the space of continuous functions on the unit circle which extend to continuous functions on $\overline{D}$ which are analytic on D (both X and Y considered as NVS with the sup norm).

   Show that X and Y are Banach spaces and are isometrically isomorphic.

7. Continuation of problem 7 of HW1:

   d Show that $(\overline{X}, ||\cdot||_{\overline{X}})$ is complete.
   Hint: A Cauchy sequence in $\overline{X}$ is a sequence $x^m = [\{x_n^m\}] \in \overline{X}$ s.t.

   $$\lim_{m_1,m_2\to\infty} \lim_{n\to\infty} ||x_n^{m_1} - x_n^{m_2}|| = 0 \text{ as } m_1, m_2 \to \infty.$$

   Find the limit of the Cauchy sequence by a diagonal argument: $y_m = x_{k_m}^m$ where k_m is chosen appropriately.

   e Show that the map $h : X \to \overline{X}$, defined by $x \mapsto [(x, x, x, \ldots)]$ is an isometric isomorphism onto its image which is dense in $\overline{X}$.

   f Show that h is onto iff $(X, ||\cdot||)$ is a Banach space.

8. Let Ω be a metric space. Recall that

   $$C_0(\Omega) = \{continuous\ f : \Omega \to R :,\ \forall\epsilon > 0\ \exists \text{ compact } L \subset \Omega :\ \forall x \notin L\ |f(x)| < \epsilon\}$$

   is a Banach space with the sup norm.

   (a) Show that

   $$C_0(\Omega) = \{continuous\ f : \Omega \to \mathbb{R} :\ \forall\epsilon > 0,\ f^{-1}((-\epsilon, \epsilon)^c)\ is\ compact\}$$

   (b) Let

   $$K_0(\Omega) = \{bounded, continuous\ f : \Omega \to \mathbb{R} : \lim_{x\to\infty} f(x) = 0\},$$

   where $x \to \infty$ means that $d(x, p) \to \infty$ for any fixed $p \in \Omega$.

   Show that $(K_0(\Omega), ||\cdot||_{\text{sup}})$ is a Banach space.

   (c) Show that $C_0(\Omega) \subset K_0(\Omega)$.

(d) Give an example to show that $K_0(\Omega)$ need not be contained in $C_0(\Omega)$.

Hint: Define a continuous function on an infinite dimensional NVS Ω which is constant on the unit ball and decays to 0 as $||x|| \to \infty$.

9. (a) Let $X = Y = c_0$, the set of sequences that converge to 0, and let $T : X \to Y$ be defined by

$$T(\{x_n\}) = \{x_n(1 - 1/n)\}$$

Show that T is a BLT with $||T|| = 1$, but for any x in the unit sphere, $||Tx|| < 1$.

(b) Let X be a reflexive Banach space and $Y = K$. Show that if $T : X \to Y$ is a BLF with $||T|| = 1$, then there exists x in the unit sphere of X s.t. $||Tx|| = 1$.