Math 421/510 Homework 1: Due Friday, January 18, 11AM. SOLUTIONS

1. (a) Show that a subset of a complete metric space is complete iff it is closed.
   (b) Show that every compact metric space is complete.
   (c) Show that for a subset S of a compact metric space, the following are equivalent.
      i. S is compact
      ii. S is complete
      iii. S is closed
   (d) Show that on a metric space the uniform limit of continuous functions is continuous

Solution: Throughout the solutions we take X to be our ambient metric space and S will denote some subset.

a.

If:

Any Cauchy sequence in S is, of course, a Cauchy sequence in X and thus converges to some $x \in X$. If S is closed then $x \in S$ and so the sequence converges to an element of S. Thus, S is complete.

Only If:

Let $x \in X$ be a limit point of S. Then x is the limit of a sequence y_n in S and thus is Cauchy in S. Since S is complete, y_n must converge to some $y \in S$. But y_n converges to x. So, $x = y \in S$. Thus, S is closed. $\square$

Note: in the "Only if" we do not need the ambient space to be complete.

b. Let x_n be Cauchy in compact X. Then x_n has a convergent subsequence to some $x \in X$. Thus, x_n converges to x. So X is complete.

c. Assume compact. Then S is a compact metric space in its own right. So it is complete by part b.

Assume complete. By part a, it is closed.

Assume closed. Let x_n be a sequence in S. Since X is compact it has a subsequence converging to a point $x \in X$. Since S us closed $x \in S$. Thus, S is compact.

d. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from X to Y, both metric spaces, converging uniformly to f. Let us show f is continuous.

Let $x \in X$ and let $\varepsilon > 0$. We need to show that there is $\delta > 0$ such that if $0 < d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

Note that for all $n \in \mathbb{N}$ the triangle inequality gives us.

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), f_n(y)) + d_Y(f_n(y), f(y)) \tag{1}$$

By uniform convergence, pick n such that $\sup_{x\in X} d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$. Since f_n is continuous, pick $\delta > 0$ such that $d_Y(f_n(x), f_n(y)) < \frac{\varepsilon}{3}$ whenever $d_X(x, y) < \delta$.

Putting everything together in (1) we get that if $d_X(x, y) < \delta$, then

$$d_Y(f(x), f(y)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

as desired.

2. (a) Show that the collection of unions of open balls in a metric space is a topology.

   (b) Show that the topology of the discrete metric is the collection of all subsets.

   (c) Show that the trivial topology, $\mathcal{T} = \{\emptyset, X\}$ on a set X with at least two points does not come from a metric.

Solution:

a.

- ($\emptyset, X \in \mathcal{B}$). We have $\emptyset = \bigcup_{x\in\emptyset} B_1(x)$ or even $\emptyset = B_0(x)$ for any $x \in X$ (if $X \neq \emptyset$). Also, $X = \bigcup_{r>0} B_r(0)$ or even $X = \bigcup_{x\in X} B_1(x)$.
- Stable under unions. This is immediate by definition of $\mathcal{B}$ as unions of balls.
- Stable under finite intersection. Let $x_1, \ldots, x_n \in X$ and $r_1, \ldots, r_n > 0$. Let $I = \bigcap_{i=1}^{n} B_{r_i}(x_i)$. For each $x \in I$ we have

$$B_{s_x}(x) \subset I \qquad \text{where} \qquad s_x = \min_{1\leq i\leq n} (r_i - d(x, x_i)) > 0.$$

  Indeed, if $y \in B_{s_i}(x)$ then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) \leq s_x + d(x, x_i) \leq r_i - d(x, x_i) + d(x, x_i) = r_i,$$

  so $y \in B_{r_i}(x_i)$. We conclude that $I = \bigcup_{x\in I} B_{s_x}(x)$.

b. For the discrete metric on X, note that for all $x \in X$ we have $B_{1/2}(x) = \{x\}$ so every singleton is an open ball. Therefore for any subset $S \subset X$ we have $S = \bigcup_{x\in S}\{x\}$ is a union of open balls so it is open.

c. Let $x \neq y$ in X. Then for any metric d, $d(x, y) > 0$. Let $B = B_{d(x,y)/2}(x)$. Then $x \in B, y \notin B$. So, $B_{d(x,y)/2}(x)$ is an open set which is neither $\emptyset$ nor X.

3. Let $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\arctan(x) - \arctan(y)|$ for $x, y \in \mathbb{R}$ (recall that $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$).

   (a) Show that d_2 is a metric on $\mathbb{R}$.

   (b) Show that $(\mathbb{R}, d_2)$ is not complete.

   (c) Show that $(\mathbb{R}, d_1)$ and $(\mathbb{R}, d_2)$ have the same topologies, i.e., the same collections of open sets (you may assume that tan and arctan are continuous)

Solution:

a. Let $f(x) = \arctan(x)$. Then $d_2(x, y) = d_1(f(x), f(y))$.

Positivity: $d_2(x, y) = d_1(f(x), f(y)) \geq 0$ and $|f(x) - f(y)| = d_1(f(x), f(y)) = 0$ iff $f(x) = f(y)$ iff $x = y$ since f is 1-1.

Symmetry:

$$d_2(y, x) = d_1(f(y), f(x)) = |f(y) - f(x)| = |f(x) - f(y)| = d_1(f(x), f(y)) = d_1(x, y)$$

Subaddivity:

$$d_2(x, z) = d_1(f(x), f(z)) \leq d_1(f(x), f(y)) + d_1(f(y), f(z)) = d_2(x, y) + d_2(y, z)$$

b. Let $x_n = n$ for all $n \in \mathbb{N}$. Since $\arctan(x_n)$ is a convergent sequence with $\lim_{n\to\infty} \arctan(x_n) = \frac{\pi}{2}$. Therefore, $(\arctan(x_n))_{n\in\mathbb{N}}$ is Cauchy with respect to d_1 and so $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to d_2.

If x_n were converging with respect to d_2, with limit x we would have $\lim_{n\to\infty} d_2(x_n, x) = |\pi/2 - \arctan(x)| = 0$ so $\arctan(x) = \frac{\pi}{2}$, but no such x exists, so the sequence does not converge. We proved that $(\mathbb{R}, d_2)$ is not complete.

c. Let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies of the two metrics. Since $\arctan(x)$ is continuous, the inverse image of every d_2 open set is d_1-open. Thus, Thus, $\mathcal{T}_2 \subset \mathcal{T}_1$. The same argument using tan instead of arctan shows that $\mathcal{T}_1 \subset \mathcal{T}_2$.

4. Show that in a NVS, with the norm topology

   (a) $x \mapsto ||x||$ is continuous.

   (b) the closed unit ball is the closure of the open unit ball, and in particular the closed unit ball is closed.

   (c) the unit sphere is closed.

Solution:

a. Given $\epsilon > 0$, let $\delta = \epsilon$, and then by triangle inequality, if $||x - y|| < \delta = \epsilon$, then $|||x|| - ||y||| \leq ||x - y|| < \epsilon$.

b. If $x_n \in B$ then $||x_n|| < 1$ and so by a, if $x_n \to x$, then $||x|| \leq 1$. Conversely if $x \in \overline{B}$, then $x_n = x(1 - 1/n) \in B$ and $x_n \to x$.

c. Call S the unit sphere, then $S = \overline{B_1(0)} \backslash B_1(0)$, a closed set minus an open set is closed. Another way to see it is that $S^c = \{x : \|x\| \neq 1\} = B_1(0) + \bigcup_{\|x\|>1} B_{\|x\|-1}(x)$ which is open.

A third way to see it is that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in S, converging to x in the ambient space, then by continuity of the norm, we have

$$\|x\| = \lim_{n\to\infty} \|x_n\| = 1.$$

5. (a) Show that the Euclidean norm and ℓ_1 norm on $\mathbb{R}^n$ and $\mathbb{C}^n$ are norms.
(b) Show that two norms on the same vector space have the same topology, i.e., same collections of open sets, iff they are equivalent as norms.
(c) Show that for two normed vector spaces with the same topology, one is complete iff the other is.
(d) Compare the results of problems 3 and 5c. What does this tell you?

Solution:

a.

Eucldean norm: $||x||_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$

Positivity: Clearly $||x||_2 = 0$ iff each $x_i = 0$.

Homogeneity:

$$||\lambda x||_2 = \sqrt{\sum_{i=1}^n |\lambda x_i|^2} = |\lambda|\sqrt{\sum_{i=1}^n |x_i|^2} = ||x||_2$$

Triangle inequality:

$$||x+y||_2^2 = \sum_{i=1}^n |x_i + y_i|^2 = \sum_{i=1}^n |x_i|^2 + \sum_{i=1}^n |y_i|^2 + 2Re(\sum_{i=1}^n x_i \overline{y_i})$$

$$\leq \sum_{i=1}^n |x_i|^2 + \sum_{i=1}^n |y_i|^2 + 2\sqrt{(\sum_{i=1}^n |x_i|^2)(\sum_{i=1}^n |y_i|^2)} = (||x||_2 + ||y||_2)^2$$

where the inequality follows from the Cauchy-Schwartz inequality.

ℓ_1 norm:

Positivity: Clearly $||x||_1 = 0$ iff each $x_i = 0$.

Homogeneity:

$$||\lambda x||_1 = \sum_{i=1}^n |\lambda x_i| = |\lambda \sum_{i=1}^n |x_i| = |\lambda| \; ||x||_1$$

Triangle inequality:

$$||x+y||_1 = \sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = ||x||_1 + ||y||_1$$

b. Let $B_\epsilon^i(x)$ be the ball in metric i.

"If:" We show that any ball in metric 2 is a union of balls in metric 1.

Let $y \in B_{\epsilon_2}^2(x)$. Let $0 < \epsilon_1 < (1/C_2)(\epsilon_2 - ||x-y||_2)$.

If $z \in B^1_{\epsilon_1}(y)$, then

$$||z-x||_2 \le ||z-y||_2 + ||y-x||_2 \le C_2||z-y||_1 + ||y-x||_2 < C_2\epsilon_1 + ||y-x||_2 < \epsilon_2$$

So, $B^1_{\epsilon_1}(y) \subset B^2_{\epsilon_2}(x)$,

So, $\mathcal{T}_2 \subset \mathcal{T}_1$. Reversing the roles of metrics 1 and 2, we get $\mathcal{T}_1 \subset \mathcal{T}_2$.

"Only if:"

If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $B^1_1(0)$ is the union of open balls in norm 2. In particular, for some x and $\epsilon > 0$,

$$0 \in B^2_\epsilon(x) \subset B^1_1(0)$$

But then letting $\delta = \epsilon - d_2(0,x)$, for any $y \in B^2_\delta(0)$, we have

$$d_2(y,x) \le d_2(y,0) + d_2(0,x) < (\epsilon - d_2(0,x)) + d_2(0,x) = \epsilon,$$

and so

$$B^2_\delta(0) \subset B^2_\epsilon(x)$$

and so

$$B^2_\delta(0) \subset B^1_1(0)$$

By continuity of $||\cdot||$, is $||x||_2 \le \delta$, then $||x||_1 \le 1$.

c. This follows from part b and the fact that two equivalent norms have the same Cauchy sequences and convergent sequences.

d. Part c is true for NVS but false for metric spaces in general.

6. Show that on a finite-dimensional vector space any two norms are equivalent as norms. In particular, the norm metric for any finite-dimensional vector space is complete.

Solution.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Let $M = \sum_{i=1}^n \|e_i\|$. Write arbitrary $x = \sum_1^n \; a_i e_i$.

$$\|x\| = \left\| \sum_1^n \; a_i e_i \right\| \le \sum_1^n |a_i| \; \|e_i\| \le M \max_i |a_i| = M\|x\|_{\text{sup}}$$

This proves half of the equivalence and that $x \mapsto \|x\|$ is Lipschitz cts. w.r.t. the metric $d(x,y) = \|x-y\|_{\text{sup}}$. Thus, $\|x\|$ achieves a minimum on the unit ball in the sup norm, say on some point u and this minimum cannot be 0 since the unit ball in any norm does not contain the origin. Thus, for all $x \neq 0$

$$\|x\| = \|x\|_{\text{sup}} \left\| \frac{x}{\|x\|_{\text{sup}}} \right\| \ge \|u\| \; \|x\|_{\text{sup}}$$

and so for all x

$$\|x\| \ge \|u\| \; \|x\|_{\text{sup}}$$

and this proves the other half of the equivalence.

The second statement in the problem follows from the first since $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete in the Euclidean norm and therefore complete in all norms.

7. Let $(X, ||\cdot||)$ be a NVS. For Cauchy sequences $\{x_n\}, \{y_n\}$ define the relation $\{x_n\} \sim \{y_n\}$ if $||x_n - y_n|| \to 0$.

   Let $\overline{X}$ denote the equivalence classes of $\sim$.

   (a) Show that $\sim$ is indeed an equivalence relation

   (b) Define vector addition ($[\{x_n\}]+[\{y_n\}] = [\{x_n+y_n\}]$) and scalar multiplication ($\lambda[\{x_n\}] = [\{\lambda x_n\}]$) on $\overline{X}$ and verify that it is well-defined and that $\overline{X}$ is a vector space with these operations.

   (c) Define $||[\{x_n\}]||_{\overline{X}} := \lim_{n\to\infty} ||x_n||$. Show that the limit exists and is a well-defined norm on $\overline{X}$.

   To be continued in HW2; can you guess where we are headed?

Solution:

a. Only transitivity is non-obvious. Let $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. Then

$$||x_n - z_n|| \leq ||x_n - y_n|| + ||y_n - z_n|| \to 0.$$

b. Addition is well-defined: If $\{x_n\} \sim \{y_n\}$ and $\{x'_n\} \sim \{y'_n\}$, then

$$||x_n + y_n - x'_n - y'_n|| \leq ||x_n - x'_n|| + ||y_n - y'_n|| \to 0$$

Scalar multiplication is well-defined: If $\{x_n\} \sim \{y_n\}$, then

$$||\lambda x_n - \lambda y_n|| = |\lambda|||x_n - y_n|| \to 0$$

The vector space properties are clear (they follow from the the well-definedness we just checked and the fact that the space of Cauchy sequence is a vector space).

c.

$$|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\|$$

and so if x_n is Cauchy then so is $\|x_n\|$ (as a sequence in $\mathbb{R}$) and so it converges.

It is well-defined because if $\{x_n\} \sim \{y_n\}$ then

$$|\|x_n\| - \|y_n\|| \leq \|x_n - y_n\| \to 0$$

and so $||x_n||$, $||y_n||$ converge to the same limit.

Check that it is a norm:

i. If $\lim_n ||x_n|| = 0$ then $(0, 0, \ldots) \sim (x_n)$ and so $[(x_n)] = 0$.

ii. $\lim_n ||\lambda x_n|| = \lambda \lim_n ||x_n||$

iii.

$$\begin{aligned}\|[(x_n + y_n)]\| &= \lim_n \|x_n + y_n\| \leq \lim_n \|x_n\| + \|y_n\| \\ &= \lim_n \|x_n\| + \lim_n \|y_n\| = \|[(x_n]\| + \|[(y_n)]\|\end{aligned}$$

8. Show that for all $p < 1$ and $n \geq 2$, $(\mathbb{R}^n, \|\cdot\|_p)$ is not an NVS.

   Solution:

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$$

   For $p < 0$, $\|x\|_p = \infty$ when some $x_i = 0$.

   For $p = 0$, $\|x\|_0 = n^\infty$.

   So, when $p \leq 0$, $\|x\|_p$ is not real-valued.

   So, we assume $0 < p < 1$.

   Let $x = (1, 0, \ldots, 0)$, $x = (0, 1, 0, \ldots, 0)$. Then

$$\|x + y\|_p = ((1 + 1 + 0 + \ldots + 0))^{1/p} = 2^{1/p} > 2$$

   And

$$\|x\|_p = \|y\|_p = 1.$$

   So,

$$\|x + y\|_p > \|x\|_p + \|y\|_p$$

9. Show that for a σ-finite measure space (X, μ), $(L^\infty(X, \mu).||\cdot||_\infty)$ is a Banach space by modifying the proof, given in class, that $(B(X), ||\cdot||_{\sup})$ is a Banach space.

   Solution: For a given Cauchy sequence f_n in L^∞, let

$$X_n = \{x : |f_n(x)| > ||f_n||_\infty\}$$

   For each m, n, let

$$X_{m,n} = \{x : |f_n(x) - f_m(x)| > ||f_n - f_m||_\infty\}$$

   Let $E = (\cup_n X_n) \cup (cup_{m,n} X_{m,n})$.

   Then $\mu(E) = 0$ and for each $x \notin E$,

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$$

   Since $\mathbb{R}$ is complete, for each $x \notin E$, $f_n(x)$ converges to some number which we call $f(x) \in R$.

   Will show that $f \in L^\infty(X)$ and $||f - f_n|| \to 0$.

   Claim 1: f is essentially bounded and so $f \in L^\infty$.

   Proof: For $x \notin E$,

$$|f(x)| = \lim_n |f_n(x)| \leq \sup_n |f_n(x)| \leq \sup_n ||f_n||_\infty < \infty.$$

   Claim 2: f_n converges to f in L^∞.

For $x \notin E$,

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \le \limsup_{m\to\infty} ||f_n - f_m||_\infty$$

Thus,

$$||f_n - f||_\infty \le \limsup_{m\to\infty} ||f_n - f_m||_\infty$$

Thus,

$$\lim_{n\to\infty} ||f_n - f||_\infty \le \lim_{n\to\infty} \limsup_{m\to\infty} ||f_n - f_m||_\infty = 0. \square$$

10. (a) Show that for $f_n, f \in L^\infty$, f_n converges to f in L^∞, i.e., $||f_n - f||_\infty \to 0$, iff $f_n \to f$ uniformly off a set of measure zero.

    (b) Show that if (X, μ) is a finite measure space and f is a bounded measurable function, then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

Solution.

a. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions. Let E_n a set of measure 0 such that $\|f_n - f\| = \sup_{x\notin E} |f_n(x) - f(x)|$, namely $E_n = \{x : |f_n(x) - f(x)| > \|f_n - f\|\}$. Then $E = \bigcup_{n\in\mathbb{N}} E_n$ is a set of measure 0, as a countable union of sets of measure 0. By construction, $\|f_n - f\| = \sup_{x\notin E} |f_n - f| \to 0$, so $f_n \to f$ uniformly off E. Conversely, if $f_n \to f$ uniformly off a set S of measure 0, then $0 \le \|f_n - f\| \le \sup_{x\notin S} |f_n - f| \to 0$, as desired.

b. Let (X, μ) be a finite measure space and let f be a bounded measurable function. Let $E = \{x : |f(x)| > \|f\|_\infty\}$, this is a set of mesure 0, and we have

$$\begin{aligned}
\|f\|_p = \left(\int_X |f(x)|^p \,\mathrm{d}\mu\right)^{1/p} &= \left(\int_E |f(x)|^p \,\mathrm{d}\mu\right)^{1/p} \\
&\le \left(\int_X \|f\|_\infty^p \,\mathrm{d}\mu\right)^{1/p} = (\|f\|_\infty^p \mu(X))^{1/p} \\
&= \|f\|_\infty \underbrace{\mu(X)^{1/p}}_{\to 1} \to \|f\|_\infty.
\end{aligned}$$

For the reverse inequality, fix $\varepsilon > 0$, and let $F = \{x \in X : \|f\|_\infty \ge |f(x)| \ge \|f\|_\infty - \varepsilon\}$. By definition of $\|f\|_\infty$ it is clear that F has a nonzero measure. We get

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f(x)|^p \,\mathrm{d}\mu\right)^{1/p} \\
&\ge \left(\int_F |f(x)|^p \,\mathrm{d}\mu\right)^{1/p} \\
&\ge \left(\int_F (\|f\|_\infty - \varepsilon)^p \,\mathrm{d}\mu\right)^{1/p} = ((\|f\|_\infty - \varepsilon)^p \mu(F))^{1/p} \\
&= (\|f\|_\infty - \varepsilon) \underbrace{\mu(F)^{1/p}}_{\to 1} \to (\|f\|_\infty - \varepsilon).
\end{aligned}$$

This is true for every $\varepsilon > 0$, so

$$\|f\|_\infty = \sup_{\varepsilon > 0}(\|f\|_\infty - \varepsilon) \leq \lim_{p \to \infty} \|f\|_p \leq \|f\|_\infty,$$

as desired.