Math 421/510 Homework 1: Due Friday, January 18, 11AM.

1. (a) Show that a subset of a complete metric space is complete iff it is closed.
   (b) Show that every compact metric space is complete.
   (c) Show that for a subset S of a compact metric space, the following are equivalent.
      i. S is compact
      ii. S is complete
      iii. S is closed
   (d) Show that on a metric space the uniform limit of continuous functions is continuous

2. (a) Show that the collection of unions of open balls in a metric space is a topology.
   (b) Show that the topology of the discrete metric is the collection of all subsets.
   (c) Show that the trivial topology, $\mathcal{T} = \{\emptyset, X\}$ on a set X with at least two points does not come from a metric.

3. Let $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\arctan(x) - \arctan(y)|$ for $x, y \in \mathbb{R}$ (recall that $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$).
   (a) Show that d_2 is a metric on $\mathbb{R}$.
   (b) Show that $(\mathbb{R}, d_2)$ is not complete.
   (c) Show that $(\mathbb{R}, d_1)$ and $(\mathbb{R}, d_2)$ have the same topologies, i.e., the same collections of open sets (you may assume that tan and arctan are continuous)

4. Show that in a NVS, with the norm topology
   (a) $x \mapsto ||x||$ is continuous.
   (b) the closed unit ball is the closure of the open unit ball, and in particular the closed unit ball is closed.
   (c) the unit sphere is closed.

5. (a) Show that the Euclidean norm and ℓ_1 norm on $\mathbb{R}^n$ and $\mathbb{C}^n$ are norms.
   (b) Show that two norms on the same vector space have the same topology, i.e., same collections of open sets, iff they are equivalent as norms.
   (c) Show that for two normed vector spaces with the same topology, one is complete iff the other is.
   (d) Compare the results of problems 3 and 5c. What does this tell you?

6. Show that on a finite-dimensional vector space any two norms are equivalent as norms. In particular, the norm metric for any finite-dimensional vector space is complete.

7. Let $(X, ||\cdot||)$ be a NVS. For Cauchy sequences $\{x_n\}, \{y_n\}$ define the relation $\{x_n\} \sim \{y_n\}$ if $||x_n - y_n|| \to 0$.

   Let $\overline{X}$ denote the equivalence classes of $\sim$.

   (a) Show that $\sim$ is indeed an equivalence relation

   (b) Define vector addition ($[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$) and scalar multiplication ($\lambda[\{x_n\}] = [\{\lambda x_n\}]$) on $\overline{X}$ and verify that it is well-defined and that $\overline{X}$ is a vector space with these operations.

   (c) Define $||[\{x_n\}]||_{\overline{X}} := \lim_{n\to\infty} ||x_n||$. Show that the limit exists and is a well-defined norm on $\overline{X}$.

   To be continued in HW2; can you guess where we are headed?

8. Show that for all $p < 1$ and $n \geq 2$, $(\mathbb{R}^n, ||\cdot||_p)$ is not an NVS.

9. Show that for a σ-finite measure space (X, μ), $(L^\infty(X, \mu).||\cdot||_\infty)$ is a Banach space by modifying the proof, given in class, that $(B(X), ||\cdot||_{\text{sup}})$ is a Banach space.

10. (a) Show that for $f_n, f \in L^\infty$, f_n converges to f in L^∞, i.e., $||f_n - f||_\infty \to 0$, iff $f_n \to f$ uniformly off a set of measure zero.

    (b) Show that if (X, μ) is a finite measure space and f is a bounded measurable function, then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.