Math 342
Homework Assignment #1 (Due Thursday, January 21, in class)
All numbered problems are worth the same value.

1. How many errors can each of the following codes correct? How many can each detect?

   (a) $\{000000, 111111, 000111\}$, $q = 2$

   (b) $\{11000, 10101, 01110, 00011\}$, $q = 2$

   (c) $\{012345, 123450, 234501, 345012, 450123\}$, $q = 6$

2. Do there exist binary codes with the following (n, M, d) parameters? (for each, say why or why not?) (5,3,5), (5,16,2), (5,17,2)

3. Consider the binary symmetric channel with channel error probability $= p$. Let C be the binary 6-repetition code.

   (a) How many errors can be corrected/detected?

   (b) If C is used as an error-correcting code with incomplete nearest-neighbour decoding, what is the probability that it will mis-correct (i.e., that it will either decode to an incorrect codeword or will declare an error when there was no error)?

   (c) If C is used as an error-detecting code, what is the probability that it will mis-detect (i.e., that it will declare an error when there was no error or declare that there is no error when there was an error)?

4. Let C be a binary code of length n to be used over the binary symmetric channel with channel error probability $= p$.

   For a codeword $\overline{c} = c_1 \ldots c_n \in C$ and a word $\overline{x} = x_1 \ldots x_n$, we define the probability that $\overline{x}$ was received, given that $\overline{c}$ was transmitted as

$$P(\overline{x}|\overline{c}) = \prod_{i=1}^{n} u_i$$

where $u_i = p$ if $x_i \neq c_i$ and $u_i = 1 - p$ if $x_i = c_i$.

(a) Show that

$$P(\overline{x}|\overline{c}) = p^d(1-p)^{n-d}$$

where $d = d(\overline{x}, \overline{c})$.

(b) In maximum likelihood decoding, the decoder decodes $\overline{x}$ to the codeword $\overline{c}$ which maximizes $P(\overline{x}|\overline{c})$ (if there is a tie, the decoder chooses an arbitrary such $\overline{c}$).

Show that if $p < 1/2$, then maximum likelihood decoding is the same as complete nearest neighbour decoding.

5. Let C be the binary 4-repetition code. For each of the received words $\overline{x} = 0000, 1000, 1100, 1110, 1111$, say what each of the following decoders will return (i.e., which codeword it will decode to or if it will declare an error)

(a) An incomplete nearest neighbour decoder which corrects 1 error.

(b) A decoder which detects up to 3 errors.

(c) A hybrid decoder that is guaranteed to correct 1 error and detect 2 errors.

6. The following are converses of results that we proved in class. Let C be a code with $|C| \geq 2$.

(a) Show that if C is u-error-detecting, then $d(C) \geq u + 1$.

(b) Show that if C is v-error-correcting, then $d(C) \geq 2v + 1$.

7. Show that for all $n \geq 2$, $A_2(n, 2) = 2^{n-1}$ and give an explicit example of a code that achieves $A_2(n, 2)$.

8. Show that for all $n \geq 4$, $A_2(n, n-1) = 2$ and give an explicit example of a code that achieves $A_2(n, n-1)$. What are $A(3, 2)$ and $A(2, 1)$?