This test consists of six pages (pages 2, 4, 6 are blank). A table of Laplace transforms is on page 7, and anything written on page 7 will not be marked.

This test consists of 3 problems, each worth 15 marks.

Time: 50 minutes. This is a closed book examination: no books, notes, electronic calculation, memory or communication devices are allowed. Calculators and cell phones are not allowed.

JUSTIFY ALL ANSWERS. Numerical answers should be “calculator-ready”.

Marks

1. An LRC series circuit is forced with an AC voltage, with \( L = 1 \) henry, \( R = 2 \) ohms, \( C = \frac{1}{5} \) farad, and voltage \( v(t) = \sin 2t \) volts. The charge \( q(t) \) on the capacitor, in coulombs, satisfies the differential equation

\[
q'' + 2q' + 5q = \sin 2t.
\]

(a) Find the general solution \( q(t) \) of the non-homogeneous equation.

(b) Identify the part(s) of the general solution that approaches 0 as \( t \to \infty \), called the transient solution.

(c) The part(s) of the general solution that is not the transient solution is the steady-state solution \( q_{ss}(t) \). If the steady-state solution is written in the phase-amplitude form \( q_{ss}(t) = Q_0 \cos(\omega t - \theta) \), what numerical values are \( Q_0 \), \( \omega \) and \( \theta \)?

Solution:

a. The characteristic equation of the homogeneous equation is \( r^2 + 2r + 5 = 0 \), whose roots are \( r = -1 \pm 2i \). So, the general solution of the homogeneous equation is

\[
c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t).
\]

The non-homogeneous term is not of this form. So, according to the method of undetermined coefficients we guess a particular solution (which is the steady-state solution) of the form

\[
y = A \cos(2t) + B \sin(2t)
\]

Plugging this into the differential equation, we obtain

\[
-4A \cos(2t) - 4B \sin(2t) + 2(-2A \sin(2t) + 2B \cos(2t)) + 5(A \cos(2t) + B \sin(2t)) = \sin(2t)
\]

which simplifies to

\[
(-4A + 4B + 5A) \cos(2t) + (-4B - 4A + 5B) \sin(2t) = \sin(2t)
\]

and results in the system of linear equations

\[
A + 4B = 0, \quad B - 4A = 1
\]
whose solution is

\[ A = -\frac{4}{17}, \quad B = \frac{1}{17}. \]

So a particular solution is:

\[ (-\frac{4}{17}) \cos(2t) + (\frac{1}{17}) \sin(2t). \]

The general solution of the non-homogeneous equation is

\[ c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) + (-\frac{4}{17}) \cos(2t) + (\frac{1}{17}) \sin(2t). \]

b. \[ c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) \]

c. \[ (-\frac{4}{17}) \cos(2t) + (\frac{1}{17}) \sin(2t), \]

which can be written in the form \( Q_0 \cos(\omega t - \theta) \), where

\[ Q_0 = \sqrt{(-\frac{4}{17})^2 + (\frac{1}{17})^2} = \frac{\sqrt{17}}{17}, \]

\[ \omega = 2, \]

\[ \theta = \arctan(-\frac{1}{4}) + \pi = -\arctan(\frac{1}{4}) + \pi \]

(noting that \((-\frac{4}{17}, \frac{1}{17})\) is in quadrant II).
2. Solve the initial value problem

\[ y'' + y = g(t) + \delta(t - 5), \quad y(0) = 0, y'(0) = 0 \]

where

\[ g(t) = \begin{cases} 
2 & 0 \leq t < 3 \\
0 & t \geq 3 
\end{cases} \]

You may leave your answer in terms of unit step functions.

**Solution:**

\[ g(t) = 2 - 2u_3(t). \]

From the table, we obtain

\[ \mathcal{L}(g(t)) = \frac{2}{s} - \frac{2e^{-3s}}{s} \quad \text{and} \quad \mathcal{L}(\delta(t - 5)) = e^{-5s}. \]

\[ \mathcal{L}(y'' + y) = s^2Y(s) + Y(s) \] where \( Y(s) = \mathcal{L}(y(t)) \). So,

\[ Y(s) = 2 \frac{1}{s((s^2 + 1)s)} (1 - e^{-3s}) + \frac{e^{-5s}}{s^2 + 1} \]

By partial fractions decomposition,

\[ \frac{1}{(s^2 + 1)s} = \frac{-s}{s^2 + 1} + \frac{1}{s} \]

So,

\[ Y(s) = \frac{-2s}{s^2 + 1} + \frac{2}{s} + \frac{2se^{-3s}}{s^2 + 1} - \frac{2e^{-3s}}{s} + \frac{e^{-5s}}{s^2 + 1} \]

From the table, we obtain

\[ y(t) = -2 \cos(t) + 2 + 2u_3(t) \cos(t - 3) - 2u_3(t) + u_5(t) \sin(t - 5) \]
(blank page for work)
3. Consider the system:

\[ x' = \begin{pmatrix} 1 & 3 \\ 0 & -3 \end{pmatrix} x \]

(a) Find a fundamental set of solutions and verify that your set is a fundamental set of solutions.

(b) Sketch the phase plane.

(c) For what values of \( a \) and \( b \), does the solution with initial conditions \( x_1(0) = a, x_2(0) = b \), converge to \((0, 0)\) as \( t \to \infty \)?

**Solution:**

Characteristic equation:

\[
\det \begin{bmatrix} 1 - r & 3 \\ 0 & -3 - r \end{bmatrix} = 0.
\]

which is \( r^2 + 2r - 3 = 0 \), which factors as \((r - 1)(r + 3) = 0\).

So, the eigenvalues are \( r = 1, -3 \).

An eigenvector for \( r = 1 \) is found by solving

\[
\begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]

which can be written as

\[ v_1 + 3v_2 = v_1, -3v_2 = v_2 \]

It follows that \( v_2 = 0 \) and \( v_1 \) can be taken to be any non-zero real number. We choose

\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

So, one solution is

\[
\begin{bmatrix} e^t \\ 0 \end{bmatrix}
\]

An eigenvector for \( r = -3 \) is found by solving

\[
\begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -3w_1 \\ -3w_2 \end{bmatrix}
\]

which can be written as

\[ w_1 + 3w_2 = -3w_1, -3w_2 = -3w_2 \]

which is equivalent to \( w_2 = -(4/3)w_1 \).

We choose

\[
\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}
\]

So, another solution is

\[
\begin{bmatrix} 3e^{-3t} \\ -4e^{-3t} \end{bmatrix}
\]
A fundamental set of solutions is:

\[
\left\{ \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \begin{bmatrix} 3e^{-3t} \\ -4e^{-3t} \end{bmatrix} \right\}
\]

because the wronskian is

\[
\det \begin{bmatrix} e^t & 3e^{-3t} \\ 0 & -4e^{-3t} \end{bmatrix} = -4e^{-2t} \neq 0.
\]

b. Draw a saddle with expanding direction

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

and contracting direction

\[
\begin{bmatrix} 3 \\ -4 \end{bmatrix}
\]

c. The origin is a saddle equilibrium and so the only solutions that converge to the origin, as \( t \to \infty \), are those with initial conditions along the eigenvector corresponding to the negative eigenvalue. These initial conditions are \( b = -(4/3)a \), where \( a \) is arbitrary.