Lecture 34:
Teaching evaluations: open until Monday, Dec. 7.
HW10 posted, but not to be turned in. Will discuss on Friday.
TA comments on HW papers is posted on the website.

Recall: Heat equation:
\[
\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.
\]

In different notation:
\[
\alpha^2 u_{xx} = u_t.
\]

Recall 3. (of three boundary value problems):
Insulated boundary conditions: \( u_x(0, t) = 0, u_x(L, t) = 0, t > 0 \).
Initial condition: \( u(x, 0) = f(x) \)
Apply same separation of variables approach: \( u(x, t) = X(x)T(t) \).
\[
X'' + \lambda X = 0, T' + \alpha^2 \lambda T = 0
\]
The boundary conditions imply \( X'(0) = 0, X'(L) = 0 \).
We found that the only possible \( \lambda \) for which there exists a solution to the \( X \)-equation are:
\[
\lambda_n = n^2 \pi^2 / L^2
\]
and the corresponding solutions are
\[
X(x) = k_2 \cos(n\pi x / L)
\]
where \( k_2 \) is an arbitrary constant.
Corresponding solution of the \( T \)-equation is:
\[
Ce^{-\frac{n^2 \pi^2 \alpha^2 t}{L^2}}
\]

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where $C$ is an arbitrary constant.

Thus, the solution is:

$$u(x, t) = c_0/2 + \sum_{n=1}^{\infty} c_n u_n(x, t)$$

where

$$u_n(t) = e^{-\frac{n^2\pi^2\alpha^2}{L^2}} \cos(n\pi x/L),$$

(note that we have absorbed the constants $k_2$ and $C$ into the $c_n$) and

$$c_n = (2/L) \int_0^L f(x) \cos(n\pi x/L) \, dx$$

This is a Fourier cosine series (the $c_n$ are the $a_n$).

For $n = 0$, the $n$-th term is a constant term, which can be viewed as a uniform steady-state solution, while the other terms decay to 0 exponentially fast as $t \to \infty$.

Display Figure 10.6.2:

Picture of solution at various $t$, with $L = 25$, initial condition $f(x) = x$, boundary condition $u_x(0, t) = 0, u_x(25, t) = 0$: The steady state solution is uniform:

$$c_0/2 = (1/25) \int_0^{25} x \, dx = 12.5$$

**The Wave Equation**

An elastic string stretched tightly between two fixed positions, satisfies the PDE:

$$a^2 u_{xx} = u_{tt}$$

where $u(x, t)$ is a function of position $x$ and time $t$. The function $u(x, t)$ measures the vertical displacement of the string above position $x$. 

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Boundary conditions: \( u(0, t) = 0, u(L, t) = 0 \), means that the string is fixed at \( x = 0 \) and \( x = L \).

Initial conditions: initial position and initial velocity:

\[
 u(x, 0) = f(x), \quad u_t(x, 0) = g(x)
\]

We require that \( f(0) = f(L) = 0 \) and \( g(0) = g(L) = 0 \).

Case 1: we assume that the string is displaced with an initial displacement \( u(x, 0) = f(x) \) but with zero initial velocity \( u_t(x, 0) = 0 \).

Again, apply separation of variables, i.e., guess \( u(x, t) = T(t)X(t) \). Plugging into the PDE, we get:

\[
a^2 T(t) X''(x) = T''(t) X(x).
\]

and so

\[
 \frac{X''}{X} = \frac{1}{a^2} \frac{T''}{T}
\]

and the common value must be a constant, again denoted \(-\lambda\).

\[
 X'' + \lambda X = 0, \quad T'' + a^2 \lambda T = 0.
\]

The boundary conditions \( u(0, t) = 0, u(L, t) = 0 \) translate to \( X(0) = 0, X(L) = 0 \). These are the same as the homogeneous boundary conditions for the heat equation and so, as in that problem, the only possible values (eigenvalues) of \( \lambda \) are:

\[
 \lambda_n := \frac{n^2 \pi^2}{L^2}
\]

with corresponding solutions (eigenfunctions)

\[
 X_n(x) = \sin \left( \frac{n \pi x}{L} \right)
\]
So,
\[ T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0. \]
and so
\[ T(t) = k_1 \cos\left(\frac{n\pi at}{L}\right) + k_2 \sin\left(\frac{n\pi at}{L}\right) \]

The initial condition \( u_t(x, 0) = 0 \) translates to \( T'(0) = 0 \) and so
\[ 0 = T'(0) = -k_1 \sin\left(\frac{n\pi a 0}{L}\right) + k_2 \cos\left(\frac{n\pi a 0}{L}\right) = k_2 \]
and so
\[ T(t) = k_1 \cos\left(\frac{n\pi at}{L}\right) \]

Thus, the solution is:
\[ u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) \]
where
\[ u_n(t) = \cos\left(\frac{n\pi at}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \]
and the initial condition \( u(x, 0) = f(x) \) requires that
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \]
which is a Fourier sine series, and so
\[ c_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx \]

Period in time: \( \frac{2L}{a} \).
Lecture 35:
Recall wave equation:
\[ a^2u_{xx} = u_{tt} \]
where \( u(x, t) \) is a function of position \( x \) and time \( t \).

Boundary conditions: \( u(0, t) = 0, u(L, t) = 0 \), means that the string is fixed at \( x = 0 \) and \( x = L \).

Initial conditions: initial position and initial velocity:
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \]
We require that \( f(0) = f(L) = 0 \) and \( g(0) = g(L) = 0 \).
Recall solution to Case 1: \( g(x) = 0 \) (zero initial velocity)
\[ u(x, t) = \sum_{n=1}^{\infty} c_n \cos(n\pi at/L) \sin(n\pi x/L) \]
The form for \( \sin \) followed from the boundary conditions and the form for \( \cos \) followed from the initial condition \( u_t(x, 0) = 0 \).
And the initial condition \( u(x, 0) = f(x) \) requires that
\[ f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) \]
Thus, we need to express \( f \) as a Fourier sine series, and so
\[ c_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) \, dx \]
Recall that \( f \) is defined only on \( 0 < x < L \) and if we extend it to \( -L < x < L \) as an odd function of period \( 2L \), then the Fourier series is a Fourier sine series.
Example 1, p. 647: Wave equation with $a = 2$, $L = 30$

$$f(x) = \begin{cases} \frac{x}{10} & 0 \leq x \leq 10 \\ \frac{30-x}{20} & 10 < x \leq 30 \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{2n\pi t}{30}\right) \sin\left(\frac{n\pi x}{30}\right),$$

\begin{align*}
    c_n &= \frac{2}{30} \int_0^{10} \frac{x}{10} \sin\left(\frac{n\pi x}{30}\right) dx + \frac{2}{30} \int_{10}^{20} \frac{30-x}{20} \sin\left(\frac{n\pi x}{30}\right) dx \\
    &= \frac{9}{n^2 \pi^2} \sin\left(\frac{n\pi}{3}\right)
\end{align*}

Display Figures:

Fig. 10.7.4 shows $u(x,t)$ as a function of $x$ for several values ("snapshots") of $t$.

Note that the time period of the wave is $30 = \frac{2L}{a} = \frac{2\pi}{2\pi/30}$. It moves from the top figure to the bottom figure in time 15, which is 1/2 of the time period.

Fig. 10.7.5 shows $u(x,t)$ as a function of $t$ for several values of $x$.

Fig. 10.7.6. shows the graph of $u(x,t)$ as a surface above the $xt$-plane.

You can visualize the cross-sections of Fig. 10.7.6 in Fig. 10.7.5 and Fig. 10.7.4.

Case 2: initial displacement is 0: $u(x,0) = f(x) = 0$ and there may be nonzero initial velocity.

Since the boundary conditions have not changed, we get the same values (eigenvalues) for $\lambda$

$$\lambda_n := \frac{n^2 \pi^2}{L^2}$$
and corresponding solutions (eigenfunctions)

\[ X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \]

And again

\[ T(t) = k_1 \cos\left(\frac{n\pi a t}{L}\right) + k_2 \sin\left(\frac{n\pi a t}{L}\right) \]

but the initial condition \( u(x,0) = 0 \) forces \( T(0) = 0 \) instead of \( T'(0) = 0 \).

This forces \( k_1 = 0 \) and so

\[ T(t) = k_2 \sin\left(\frac{n\pi a t}{L}\right) \]

So the solution is

\[ u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \]

Now,

\[ u_t(x,t) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L} \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \]

So,

\[ g(x) = u_t(x,0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L} \sin\left(\frac{n\pi x}{L}\right) \]

Thus, again we have a Fourier sine series where

\[ c_n = \frac{L}{n\pi a} \frac{(2/L)}{2} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \, dx = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \, dx \]

Finally, for general initial conditions

\[ u(x,0) = f(x), \quad u_t(x,0) = g(x) \]

the solution is

\[ u(x,t) = v(x,t) + w(x,t) \]

where \( v(x,t) \) is the solution for zero initial velocity and \( w(x,t) \) is the solution for zero initial displacement. □