Chapter 10: Partial differential equations and Fourier series

Heat equation: Let \( u(x, t) \) be the temperature in a thin heat-conducting bar of length \( L \) (where \( x \) is position and \( t \) is time).

\[
\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.
\]

Will use methods of Fourier series to solve such equations.

Represent \( f(x) \) as

\[
f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right))
\]

Main problems are: which functions can be represented in such a way and how can we find the “Fourier” coefficients \( a_m, b_m \)?

Defn: A function \( f \) is periodic with period \( T \) if \( f(x + T) = f(x) \) for all \( x \) (s.t. \( x \) and \( x + T \) are in the domain of \( f \)).

Defb: The fundamental period of a periodic function \( f \) is the smallest period of \( f \).

Turns out that every non-constant function has a fundamental period (the constant function is periodic for all periods \( T \)).

Fact: \( \sin(x) \) and \( \cos(x) \) are periodic with fundamental period \( 2\pi \).

Fact: \( \sin(\alpha x) \) and \( \cos(\alpha x) \) has fundamental period \( 2\pi/\alpha \).

So, \( \sin\left(\frac{m\pi x}{L}\right) \) and \( \cos\left(\frac{m\pi x}{L}\right) \) have fundamental period

\[
\frac{2\pi}{m\pi/L} = \frac{2L}{m}
\]

in particular they have period \( 2L \).
So, a function which can be represented by a Fourier series must have period $2L$.

Recall the *dot product* in $\mathbb{R}^3$.

$$ u \cdot v := u_1 v_1 + u_2 v_2 + u_3 v_3 $$

We say $\vec{u}, \vec{v}$ are orthogonal if $u \cdot v = 0$.

In $\mathbb{R}^3$, $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is mutually orthogonal.

Defn: *Inner product*: Let $u(x)$ and $v(x)$ be functions on the interval $a \leq x \leq b$.

$$ u \cdot v := \int_a^b u(x)v(x) \, dx $$

This shares many properties with the dot product on $\mathbb{R}^3$.

Defn: functions $u$ and $v$ are *orthogonal* if $u \cdot v = 0$.

A set of functions is *mutually orthogonal* if any two distinct elements of the set are orthogonal.

The key to finding the Fourier coefficients is:

Proposition: The set

$$ \{\cos(\frac{m\pi x}{L}), \sin(\frac{m\pi x}{L}), 1\} $$

of functions on the interval $-L \leq x \leq L$ is mutually orthogonal.

Example: Orthogonality between $\cos(\frac{m\pi x}{L})$ and $1$:

$$ (\cos(\frac{m\pi x}{L})) \cdot 1 = \int_{-L}^{L} \cos(\frac{m\pi x}{L}) \, dx $$

$$ \frac{L}{m\pi} \sin(\frac{m\pi x}{L})\bigg|_{-L}^{L} $$

$$ \frac{L}{m\pi} (\sin(m\pi) - \sin(-m\pi)) = 0 - 0 = 0. $$
(because $m$ is an integer).

The proof uses trig identities:

1. $\cos(\alpha) \cos(\beta) = (1/2)(\cos(\alpha - \beta) + \cos(\alpha + \beta))$

2. $\sin(\alpha) \cos(\beta) = (1/2)(\sin(\alpha - \beta) + \sin(\alpha + \beta))$

3. $\sin(\alpha) \sin(\beta) = (1/2)(\cos(\alpha - \beta) - \cos(\alpha + \beta))$

To show that the following pairs of functions on $-L \leq x \leq L$ are orthogonal:

1. \[\{\cos\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\} \text{ for } m \neq n\]

2. \[\{\sin\left(\frac{m\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\} \text{ for } m \neq n\]

3. \[\{\sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\} \text{ for all } m, n\]

For simplicity assume $L = 1$.

1. Since $m \neq n$, $m - n \neq 0$, $m + n \neq 0$,

$$\int_{-1}^{1} \cos(m\pi x) \cos(n\pi x) = \int_{-1}^{1} \cos((m-n)\pi x) + \cos((m+n)\pi x) \, dx =$$

$$(1/\pi)\left(\frac{\sin((m-n)\pi x)}{m-n} + \frac{\sin((m+n)\pi x)}{m+n}\right)|_{-1}^{1} = 0.$$ because it involves sums/differences of $\sin(k\pi)$, for integers $k$.

3. is similar.
2. is similar when \( m \neq n \). For \( m = n \),

\[
\int_{-1}^{1} \sin(m\pi x) \cos(n\pi x) = \int_{-1}^{1} \sin(0) + \sin((2n)\pi x) \, dx \\
= -(1/\pi)(1/2n) \cos(2n\pi x)|_{-1}^{1} = 0.
\]

\[\square\]

Proposition: Consider the inner product of a trig function with itself:

\[
\int_{-L}^{L} \cos(n\pi x/L) \cos(n\pi x/L) = \int_{-L}^{L} \cos^2(n\pi x/L) = L
\]

and

\[
\int_{-L}^{L} \sin(n\pi x/L) \sin(n\pi x/L) = \int_{-L}^{L} \sin^2(n\pi x/L) = L
\]
Lecture 29:
Recall:
Proposition 1: The set
\[ \{ \cos \left( \frac{m \pi x}{L} \right), \sin \left( \frac{m \pi x}{L} \right), 1 \} \]
of functions on the interval \(-L \leq x \leq L\) is mutually orthogonal.

and

Proposition 2: Consider the inner product of a trig function with itself:
\[ \int_{-L}^{L} \cos \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi x}{L} \right) \, dx = \int_{-L}^{L} \cos^2 \left( \frac{n \pi x}{L} \right) = L \]
and
\[ \int_{-L}^{L} \sin \left( \frac{n \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \, dx = \int_{-L}^{L} \sin^2 \left( \frac{n \pi x}{L} \right) \, dx = L \]

Full proofs, using trig identities, are on pages 598–599.

For Prop. 1, we only showed that \(\cos \left( \frac{m \pi x}{L} \right)\) and 1 are orthogonal:
\[ \left( \cos \left( \frac{m \pi x}{L} \right) \right) \cdot 1 = \int_{-L}^{L} \cos \left( \frac{m \pi x}{L} \right) \, dx = 0. \]

Picture of example when \(L = 1\) and \(m = 1\).

Another case: \(\cos \left( \frac{m \pi x}{L} \right)\) and \(\sin \left( \frac{n \pi x}{L} \right)\) (all \(m, n\)), are orthogonal.

Example: \(L = 1\) and \(m = n = 1\).
\[ \int_{-1}^{1} \cos(\pi x) \sin(\pi x) = (1/2) \int_{-1}^{1} \sin(2\pi x) \]
\[ = -\left( \frac{1}{4\pi} \right) \cos(2\pi x) \bigg|_{-1}^{1} = -\left( \frac{1}{4\pi} \right)(1 - 1) = 0. \]
Example for Prop. 2: \( L = 1 \) and \( n = 1 \).

\[
\int_{-1}^{1} \cos^2(\pi x) dx = \frac{1}{2} \int_{-1}^{1} (1 + \cos(2\pi x)) dx \\
= 1 + \frac{1}{2} \sin(2\pi x)\bigg|_{-1}^{1} = 1 + 0 - 0 = 1.
\]

\[\square\]

Euler-Fourier formulas: These are formulas for the Fourier coeffs.

If

\[
f(x) = a_0 + \sum_{m=1}^{\infty} \left( a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)
\]

then

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx
\]

and for \( n \geq 1 \),

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L) dx \\
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L) dx
\]

Proof: By orthogonality between the trig functions and the function 1,

\[
\int_{-L}^{L} \cos(m\pi x/L) = 0, \quad \int_{-L}^{L} \sin(m\pi x/L) = 0.
\]

For \( n \geq 1 \),

\[
\int_{-L}^{L} f(x) \cos(n\pi x/L) dx = f \cdot \cos\left(\frac{n\pi x}{L}\right) = \left(\frac{a_0}{2}\right)1 \cdot \cos\left(\frac{n\pi x}{L}\right) \\
+ \sum_{m=1}^{\infty} \left( a_m \cos\left(\frac{m\pi x}{L}\right) \right) \cdot \cos\left(\frac{n\pi x}{L}\right)
\]
\[ + \sum_{m=1}^{\infty} (b_m \sin \left( \frac{m\pi x}{L} \right) \cdot \cos \left( \frac{n\pi x}{L} \right) \]
\[ = \left( \frac{a_0}{2} \right) \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \]
\[ + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \]
\[ = L a_n \]

and so

\[ a_n = \left( \frac{1}{L} \right) \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx \]

Similarly one can determine \( b_n \).

And

\[ \int_{-L}^{L} f(x) dx = \left( \frac{a_0}{2} \right) \int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \left( \frac{m\pi x}{L} \right) \]
\[ + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \left( \frac{m\pi x}{L} \right) = L a_0 \]

\[ \square \]

Example 1:

\[ f(x) = \begin{cases} 
-x & -2 \leq x < 0 \\
0 & 0 \leq x \leq 2 
\end{cases} \]

and extend periodically \((f(x + 4) = f(x))\) to obtain a function on the real line.

So, \( L = 2 \).

\[ a_0 = \left( \frac{1}{2} \right) \int_{-2}^{2} f(x) dx = \left( \frac{1}{2} \right) \left( \int_{-2}^{0} -xdx + \int_{0}^{2} xdx \right) = 2. \]
Coeff of cos:

\[ a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos(n\pi x/2)\,dx = \frac{1}{2} \int_{-2}^{0} -x \cos(n\pi x/2) \]
\[ + \frac{1}{2} \int_{0}^{2} x \cos(n\pi x/2) \]

(integrate by parts, p. 601) \[ \int x \cos(x)\,dx = \cos(x) + x \sin(x) \]
\[ = \begin{cases} 
-8/(n\pi)^2 & \text{n odd} \\
0 & \text{n even} 
\end{cases} \]

Coeff of sin: It turns out that each \( b_n = 0 \). So,
\[ f(x) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n - 1)\pi x/2)}{(2n - 1)^2} \]

(note: \( 2n - 1 \) is odd).
Lecture 30:

Last time, we showed how to find the Fourier coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ if a function $f$ has a Fourier series.

Recall:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

then

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

Recall the defn. of piecewise continuous function:

The interval $[a, b]$ of defn. is subdivided into finitely many subintervals $a = t_0 < t_1 < \ldots < t_n = b$ s.t. $f$ is continuous on each $(t_i, t_{i+1})$ and for each $i$, both $f(x-) := \lim_{x \to t_i^-} f(x)$ and $f(x+) := \lim_{x \to t_i^+} f(x)$ exist.

Fourier Convergence Theorem (10.3.1) Suppose that $f$ and $f'$ are piecewise continuous on the interval $-L \leq x \leq L$. Suppose that $f$ is periodic with period $2L$. Then $f$ has a Fourier series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

where the series converges to

$$\begin{cases} 
    f(x) & \text{if } f \text{ is continuous at } x \\
    \frac{f(x-)+f(x+)}{2} & \text{otherwise}
\end{cases}$$
However, we can re-define $f$ at points of discontinuity to be $f(x) = \frac{f(x^-) + f(x^+)}{2}$, so that the Fourier series converges to $f(x)$ for all $x$.

Recall: Defn: $f$ is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.

Examples of even functions (p. 600): $1, x^{2n}, \cos(nx), |x|, -|x|$ and the triangular wave function

\[ f(x) = \begin{cases} -x & -2 \leq x < 0 \\ x & 0 \leq x \leq 2 \end{cases}. \]

For $x > 0$, $f(-x) = -(x) = x = f(x)$

For $x < 0$, $f(-x) = -x = f(x)$

Examples of odd functions: $x^{2n+1}, \sin(nx)$ and

Example (p. 617): The sawtooth wave function, which has discontinuities at each odd multiple of $L$:

\[ f(x) = \begin{cases} x & -L < x < L \\ 0 & x = L, -L \end{cases} \]

(extended periodically with period $2L$)

\[ f(-x) = -x = -f(x) \]

Example of function that is neither even nor odd: $e^x$.

Basic properties of even/odd functions:

1. The sum, difference, product, quotient of two even functions is even.

2. The sum and difference of two odd functions is odd; the product and quotient of two odd functions is even.

3. The product and quotient of an even function and an odd function is odd.
4. If $f$ is even, then
\[ \int_{-L}^{L} f(x) \, dx = 2 \int_{0}^{L} f(x) \, dx \]

5. If $f$ is odd, then
\[ \int_{-L}^{L} f(x) \, dx = 0. \]

Proof of 4:
\[ \int_{-L}^{L} f(x) \, dx = \int_{-L}^{0} f(x) \, dx + \int_{0}^{L} f(x) \, dx \]
Substitute $u = -x$ in first integral:
\[ \int_{-L}^{0} f(x) \, dx = -\int_{L}^{0} f(-u) \, du = -\int_{L}^{0} f(u) \, du = \int_{0}^{L} f(u) \, du \]
\[ \square \]

Proof of 5:
\[ \int_{-L}^{L} f(x) \, dx = \int_{-L}^{0} f(x) \, dx + \int_{0}^{L} f(x) \, dx \]
Substitute $u = -x$ in first integral:
\[ \int_{-L}^{0} f(x) \, dx = -\int_{L}^{0} f(-u) \, du = \int_{L}^{0} f(u) \, du = -\int_{0}^{L} f(u) \, du \]
\[ \square \]

Prop: If $f$ is even and satisfies Fourier convergence theorem, then its Fourier series is a cosine series:
\[ \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \left( \frac{m\pi x}{L} \right) \]
If $f$ is odd and satisfies Fourier convergence theorem, then its Fourier series is a sine series:

$$\sum_{m=1}^{\infty} (b_m \sin\left(\frac{m\pi x}{L}\right))$$

Proof: even case: for $n \geq 1$,

$$b_n = (1/L) \int_{-L}^{L} f(x) \sin(n\pi x/L) dx = 0$$

because $f$ is even, sin is odd and so $f \sin$ is odd.

odd case: for $n \geq 1$,

$$a_n = (1/L) \int_{-L}^{L} f(x) \cos(n\pi x/L) dx = 0$$

because $f$ is odd, cos is even and so $f \cos$ is odd. And

$$a_0 = (1/L) \int_{-L}^{L} f(x) dx = 0$$

because $f$ is odd.

Example (p 600): The triangular wave function:

Is continuous everywhere and even.

Last time, we saw:

$$f(x) = 1 - \left(\frac{8}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x/2)}{(2n-1)^2}$$

(holds for all $x$)

Example (p. 617): The sawtooth wave function,

Since $f$ is odd, each $a_n = 0$.

Since $f \sin$ is even,

$$b_n = (1/L) \int_{-L}^{L} f(x) \sin(n\pi x/L) dx = (2/L) \int_{0}^{L} f(x) \sin(n\pi x/L) dx$$
\[ = \frac{2}{L} \int_0^L x \sin(n\pi x/L) \, dx \]

(integrate by parts \( \int x \sin(ax) \, dx = (1/a)^2 (\sin(ax) - ax \cos(ax)) \))

\[ = \frac{2}{L} \left( \frac{L}{n\pi} \right)^2 \left( \sin \left( \frac{n\pi x}{L} \right) - \frac{n\pi x}{L} \cos \left( \frac{n\pi x}{L} \right) \right) \bigg|_0^L \]

\[ = \frac{2}{L} \left( \frac{L}{n\pi} \right)^2 \left( (-n\pi) \cos(n\pi) \right) = \frac{2L}{n\pi} (-1)^{n+1} \]

So

\[ f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{L} \right) \]

which is 0 at every multiple of \( L \).

A function \( f(x) \) on \( 0 < x \leq L \) can be extended to a periodic function of period \( 2L \) by extending it to a function on \( -L \leq x \leq L \) which is either odd or even or neither.

If extended to be even, then its Fourier series will be a cosine series.
If extended to be odd, then its Fourier series will be a sine series.