Recall that we are considering homogeneous linear systems with constant coefficients:

\[ \ddot{x} = A \dot{x} \]

The entries of the matrix \( A \) are constant and the entries of vector \( \dot{x} \) are functions of \( t \).

Will usually assume \( \det(A) \neq 0 \).

Recall Example 1:

\[ \ddot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \dot{x} \]

Eigenvalues of \( A \) are 2 and \(-3\) with corresponding eigenvectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), yielding solutions

\[ \dot{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} \] and \( \dot{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t} \).

The Wronskian is

\[ \det \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} = e^{-t}, \]

which is never zero.

The general solution was

\[ c_1 \dot{x}^{(1)} + c_2 \dot{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}. \]

To finish Example 2:

\[ \ddot{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \dot{x} \]
we found that the eigenvalues are 3 and $-1$. For 3, we found a corresponding eigenvector
\[
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]
and solution
\[
\vec{x}^{(1)} = \begin{bmatrix}
1 \\
2
\end{bmatrix} e^{3t}
\]
For the other eigenvalue $r = -1$, we get a corresponding eigenvector
\[
\begin{bmatrix}
1 \\
-2
\end{bmatrix}
\]
and solution
\[
\vec{x}^{(2)} = \begin{bmatrix}
1 \\
-2
\end{bmatrix} e^{-t}
\]
The Wronskian is
\[
\det \begin{bmatrix}
e^{3t} & e^{-t} \\
2e^{3t} & -2e^{-t}
\end{bmatrix} = -4e^{2t}.
\]
which is never zero. So, the general solution is:
\[
c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = c_1 \begin{bmatrix}
1 \\
2
\end{bmatrix} e^{3t} + c_2 \begin{bmatrix}
1 \\
-2
\end{bmatrix} e^{-t}.
\]
If the eigenvalues are real and distinct, this procedure will always work because the corresponding eigenvectors are linearly independent, and so the Wronskian will be non-zero.

The *direction field* of a system $\vec{x}' = A\vec{x}$ is a sketch of vectors in the plane:
\[
at \vec{x}, \text{ the vector is } A\vec{x}
\]
The *phase portrait* is the set of the solutions to the system, which are the curves tangent to the direction field.
Note: The direction field is a field of vectors, not just slopes.
Note: The independent variable $t$ is a parameter along each curve.

*Draw the direction field:* first along the eigenvectors and then “interpolate.”

*Draw the phase portrait:* find the curves tangent to the direction field.

*Equilibrium solutions:* Values of $\bar{x}$ s.t. $A\bar{x} = \bar{0}$. If you start at $\bar{x}$, then you stay at $\bar{x}$ forever.

Note: $\bar{0}$ is always an equilibrium solution.

If $\det(A) \neq 0$, then $\bar{0}$ is the only equilibrium solution.

Example 1:
$\bar{0}$ is the only equilibrium.

Example 2: (Figures 7.5.1 and 7.5.2)
$\bar{0}$ is the only equilibrium.
Lecture 24:  
Example 3:

\[ \vec{x}' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \vec{x} \]

Find characteristic equation of matrix

\[ \det \begin{bmatrix} -3 - r & \sqrt{2} \\ \sqrt{2} & -2 - r \end{bmatrix} = r^2 + 5r + 4 = (r + 4)(r + 1) \]

Eigenvalues of \( A \) are \(-1\) and \(-4\) with corresponding eigenvectors

\[ \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}, \]

yielding solutions

\[ \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} \text{ and } \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

The Wronskian is

\[ \det \begin{bmatrix} e^{-t} & -\sqrt{2} e^{-4t} \\ \sqrt{2} e^{-t} & e^{-4t} \end{bmatrix} = 3e^{-5t} \]

which is never zero.

The general solution is

\[ c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}. \]

Draw direction field along eigenlines.

\[ A \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} -1 \\ -\sqrt{2} \end{bmatrix} \]

\[ A \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ -4 \end{bmatrix} \]
Draw direction field at other point

\[
A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ \sqrt{2} \end{bmatrix}
\]
\[
A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -2 \end{bmatrix}
\]

All directions tend to point towards $\bar{0}$.

Show direction field, Fig. 7.5.3.

Draw solutions along eigenlines.

Show phase portrait, Fig. 7.5.4.

Explain that $-4$ eigenline decays to $\bar{0}$ faster than $-1$ eigenline.

So, solutions get pushed towards $-1$ eigenline.

Initial value problem: find solution $\bar{x}(t) = c_1 \bar{x}^{(1)}(t) + c_2 \bar{x}^{(2)}(t)$ s.t.

\[
\bar{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + c_2 \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}
\]

\[1 = c_1 - \sqrt{2} c_2, \quad 0 = c_1 \sqrt{2} + c_2\]

Plugging $c_2 = -c_1 \sqrt{2}$ into first equation, we get

\[1 = c_1 + 2c_1\]

So, $c_1 = 1/3, c_2 = -\frac{\sqrt{2}}{3}$. So, solution to IVP is

\[
\bar{x}(t) = (1/3) \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{-t} - \frac{\sqrt{2}}{3} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} e^{-4t}.
\]

\[
= \begin{bmatrix} (1/3)e^{-t} + (2/3)e^{-4t} \\ (1/3)\sqrt{2}e^{-t} - \frac{\sqrt{2}}{3} e^{-4t} \end{bmatrix}
\]
For real and distinct eigenvalues, $0$ is the only equilibrium point.

There are three classes of equilibrium.

**Saddle equilibrium:** The eigenvalues are of opposite sign (Examples 1 and 2)

**Stable node:** The eigenvalues are both negative (Example 3).

**Unstable node:** The eigenvalues are both positive:

$$\begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

**Complex eigenvalues:**
Recall that we guessed a solution $x = ve^{rt}$ as a solution to $x' = Ax$, and we found that $r$ must be an eigenvalue and $v$ a corresponding eigenvector.

This holds whether $r$ is real or complex.

If complex, then $r = \lambda \pm \mu i$, complex conjugates. The corresponding eigenvectors will also be complex conjugates:

$$\overline{v} = \alpha \pm \beta i$$. Then we will get two complex solutions to the system

$$x(1) = (\alpha + i\beta)e^{\lambda t}e^{i\mu t} = (\alpha + i\beta)e^{\lambda t}(\cos(\mu t) + i \sin(\mu t))$$

and

$$x^{(2)} = (\alpha - i\beta)e^{\lambda t}e^{-i\mu t} = (\alpha - i\beta)e^{\lambda t}(\cos(\mu t) - i \sin(\mu t))$$

From these we will find a fundamental set of two real solutions.
Lecture 25:

Midterm coverage: 3.5 (undet. coeff.), 3.6 (var. of parameters), 3.7/3.8 (2nd order ODE’s for circuits), 6.1-6.5 (Laplace transforms), 7.5 (2x2 systems with constant coefficients, distinct real eigenvalues), brief familiarity with sec. 7.1 and 7.4.

Laplace transform table will be provided.

Study homework solutions. As in Midterm 1, some midterm problems will be similar to homework problems.

See Nagata’s webpage for suggested problems from text.

http://www.math.ubc.ca/ nagata/m256/

Example 1, p. 408 (complex eigenvalues)

\[ A = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \]

Characteristic equation

\[
\det \begin{bmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{bmatrix} = 0
\]

\[ r^2 + r + 5/4 = 0. \]

\[ r_1 = -1/2 + i, \quad r_1 = -1/2 - i, \quad \text{with corresponding eigenvectors} \]

\[ \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix} \]

We get a fundamental set of complex solutions

\[ \bar{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1/2+i)t} \]

\[ \bar{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1/2-i)t} \]
The real and imaginary parts of either one forms a fundamental set of real solutions.

\[
\overline{x}^{(1)}(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1/2 + i)t} = e^{(-1/2)t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)(\cos(t) + i \sin(t))
\]

\[
= e^{(-1/2)t} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + ie^{(-1/2)t} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}
\]

\[
z^{(1)}(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} e^{(-1/2)t}
\]

\[
z^{(2)}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} e^{(-1/2)t}
\]

Wronskian:

\[
\det \begin{bmatrix} e^{(-1/2)t} \cos(t) & e^{(-1/2)t} \sin(t) \\ -e^{(-1/2)t} \sin(t) & e^{(-1/2)t} \cos(t) \end{bmatrix} = e^{-t} \neq 0.
\]

So, general solution is \(c_1z^{(1)}(t) + c_2z^{(2)}(t)\).

Direction field (Fig. 7.6.1) and Phase portrait (Fig. 7.6.2) show that solutions spiral into the origin. They spiral because of the \(\cos\) and \(\sin\); they move in towards the origin because of the damping factor \(e^{(-1/2)t}\).

In general, with complex eigenvalues \(r = \lambda \pm i\mu\), and corresponding eigenvectors \(\overline{a} \pm i\overline{b}\), we get solutions

\[
\overline{z}^{(1)} = (\overline{a} \cos(\mu t) - \overline{b} \sin(\mu t))e^{\lambda t}
\]

and

\[
\overline{z}^{(2)} = (\overline{b} \cos(\mu t) + \overline{a} \sin(\mu t))e^{\lambda t}
\]

and the general solution will be \(c_1\overline{z}^{(1)} + c_2\overline{z}^{(2)}\).

If \(\lambda < 0\), the solutions will spiral into the origin (damping)
If $\lambda > 0$, the solutions will spiral out from the origin (exploding). In either case, the equilibrium $\bar{0}$ is called a *spiral equilibrium*.

If $\lambda = 0$, the solutions will rotate around the origin in periodic patterns (Figures 7.6.3(b) and 7.6.4(b)) and called a *center equilibrium*.

**Repeated eigenvalues:**
Suppose that $A$ has a repeated eigenvalue $r$. Let $E = \{ \bar{v} \in \mathbb{R}^2 : A\bar{v} = r\bar{v} \}$. Then the dimension of $E$ is either 1 or 2.
If the dimension is 2, then every vector is an eigenvector.
So, for every vector $\bar{v}$, we have that $\bar{v}e^{rt}$ is a solution.
So, we can choose any convenient basis, say

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

and the general solution is

$$c_1 \begin{bmatrix} e^{rt} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{rt} \end{bmatrix} = \begin{bmatrix} c_1e^{rt} \\ c_2e^{rt} \end{bmatrix}$$

If $r < 0$, the origin will be a sink and if $r > 0$, the origin will be a source.

Example:

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Then the general solution is

$$c_1 \begin{bmatrix} e^{5t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{5t} \end{bmatrix}$$