Math 256 (Winter 2015-16) HW #9, Due on Friday, Nov. 27

1. For each function below, say whether the function is even, odd or neither.
   (a) $\sin(5x)$
   (b) $\sin(5x) + \cos(5x)$
   (c) $x^3 \cos(17x)$
   (d) $x^{2/3}$
   (e) $e^x - e^{-x}$

a. odd: $\sin(-5x) = -\sin(5x)$.

b. neither: Let $f(x) := \sin(5x) + \cos(5x)$. Then $f(-x) = \sin(-5x) + \cos(-5x) = -\sin(5x) + \cos(5x)$. So, $f(0) = 1$ and $f(-0) = f(0) = 1 \neq -1$, and so it is not odd. And $f(\pi/2) = -1$, while $f(-\pi/2) = 1$, and so is not even.

c. odd, since it is the product of an odd function and an even function.

d. even since $(-x)^{2/3} = x^{2/3}$

e. odd since $e^{-x} - e^x = -(e^x - e^{-x})$

2. Find Fourier series for each of the following functions.
   (a) $f(x) = \sin(\pi x) + \cos(3\pi x)$
   (b) $f(x) = \sin(3x)$

a. $\sin(\pi x)$ has fundamental period 2 and $\cos(3\pi x)$ has fundamental period 2/3. Since 2 is an integer multiple of 2/3, $f(x)$ has fundamental period 2. So, $L = 1$ and the Fourier series is of the form

   $$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x) + b_m \sin(m\pi x)$$

But $f(x) = \sin(\pi x) + \cos(3\pi x)$ is of this form and so it is a Fourier series of itself (one can show, using the orthogonality relations, that the Fourier series of a function is unique)

b. $\sin(3x)$ has fundamental period $2\pi/3$ and so $L = \pi/3$ and the Fourier series is of the form

   $$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(3mx) + b_m \sin(3mx)$$

But $f(x) = \sin(3x)$ is of this form and it is a Fourier series of itself.

3. Find the Fourier series for the following function and sketch the graph of the function to which the series converges over three fundamental periods.

   $$f(x) = \begin{cases} 
   0 & -2 \leq x \leq 0 \\
   2 - x & 0 < x \leq 2
   \end{cases}$$
\[ f(x + 4) = f(x). \]

Since the fundamental period is 4, \( L = 2 \). By the Euler-Fourier formulas,

\[ a_n = (1/2) \int_{-2}^{2} f(x) \cos(n \pi x / 2) = (1/2) \int_{0}^{2} (2 - x) \cos(n \pi x / 2) \]

Now,

\[ \int_{0}^{2} \cos(n \pi x / 2) = \frac{2}{n \pi} \sin(n \pi x / 2)|_{0}^{2} = 0. \]

By integration by parts, for \( n \geq 1 \),

\[ \int_{0}^{2} x \cos(n \pi x / 2) = \frac{4}{n \pi^2} (\cos(n \pi x / 2) + (n \pi / 2)x \sin(n \pi x / 2))|_{0}^{2} \]

\[ = \left( \frac{4}{n \pi^2} \right) ((-1)^n - 1) \]

Combining the previous two statements, we get for \( n \geq 1 \),

\[ a_n = \left( \frac{2}{n \pi^2} \right) (1 - (-1)^n) \]

And

\[ a_0 = (1/2) \int_{0}^{2} (2 - x) = (1/2)(2x - (1/2)x^2)|_{0}^{2} = 1 \]

And

\[ b_n = (1/2) \int_{-2}^{2} f(x) \sin(n \pi x / 2) = (1/2) \int_{0}^{2} (2 - x) \sin(n \pi x / 2) \]

Now,

\[ \int_{0}^{2} \sin(n \pi x / 2) = -\frac{2}{n \pi} \cos(n \pi x / 2)|_{0}^{2} = -\frac{2}{n \pi} ((-1)^n - 1) = \frac{2}{n \pi} (1 - (-1)^n) \]

By integration by parts

\[ \int_{0}^{2} x \sin(n \pi x / 2) = \frac{4}{n \pi^2} (\sin(n \pi x / 2) - (n \pi / 2)x \cos(n \pi x / 2))|_{0}^{2} \]

\[ = -4 \frac{2}{n \pi^2} (n \pi / 2)(-1)^n = -4 \frac{2}{n \pi} (-1)^n \]

So the Fourier series is

\[ f(x) = 1/2 + \sum_{n=1}^{\infty} \left( \frac{2}{n \pi^2} \right) (1 - (-1)^n) \cos(n \pi x / 2) + \sum_{n=1}^{\infty} \frac{2}{n \pi} \sin(n \pi x / 2) \]

4. Let \( f(x) = x - 1, \ 0 < x < 1. \)
(a) Define extensions of \( f \) to both an even function and an odd function of period 2 on \( \mathbb{R} \), and sketch both extensions.

(b) Find the Fourier series for each extension.

a. Odd extension:

\[
f(x) = \begin{cases} 
    x - 1 & 0 < x \leq 1 \\
    0 & x = 0 \\
    x + 1 & -1 \leq x < 0 
\end{cases}
\]

Even extension:

\[
f(x) = \begin{cases} 
    x - 1 & 0 < x \leq 1 \\
    -x + 1 & -1 \leq x < 0 
\end{cases}
\]

b. For odd extension, we get a Fourier sine series

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)
\]

where

\[
b_n = 2 \int_{0}^{1} (x - 1) \sin(n\pi x) = 2 \int_{0}^{1} x \sin(n\pi x) - 2 \int_{0}^{1} \sin(n\pi x) = \frac{2}{n\pi} (\sin(n\pi x) - (n\pi x)\cos(n\pi x))|_0^1 + \frac{2}{n\pi} \cos(n\pi x)|_0^1 = -\frac{2}{n\pi} (-1)^n + \frac{2}{n\pi}((-1)^n - 1) = -\frac{2}{n\pi}
\]

So,

\[
f(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin(n\pi x)
\]

For even extension, we get a Fourier cosine series

\[
f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)
\]

where, for \( n \geq 1 \),

\[
a_n = 2 \int_{0}^{1} (x - 1) \cos(n\pi x) = 2 \int_{0}^{1} x \cos(n\pi x) - 2 \int_{0}^{1} \cos(n\pi x) = \frac{2}{n^2\pi^2} (\cos(n\pi x) + (n\pi x)\sin(n\pi x))|_0^1 - \frac{2}{n\pi} \sin(n\pi x)|_0^1 = \frac{2}{n^2\pi^2}((-1)^n - 1)
\]

and

\[
a_0 = 2 \int_{0}^{1} (x - 1)dx = x^2 - 2x|_0^1 = -1
\]
So,

\[ f(x) = -1/2 + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} ((-1)^n - 1) \cos(n \pi x) \]