1. \( t^2 y'' - 2y = 0. \)

(a) To verify that a given function is a solution, we simply differentiate it and see if it and its derivative(s) satisfy the differential equation. Differentiating \( y_1(t) = t^2 \) twice, we get

\[
y_1'(t) = 2t, \quad y_1''(t) = 2,
\]

then

\[
t^2 y_1''(t) - 2y_1(t) = t^2 \cdot (2) - 2 \cdot (t^2) \equiv 0,
\]

which verifies that \( y_1(t) = t^2 \) is a solution. Doing the same for \( y_2(t) = 1/t \), we get

\[
y_2'(t) = -\frac{1}{t^2}, \quad y_2''(t) = \frac{2}{t^3},
\]

then

\[
t^2 y_2''(t) - 2y_2(t) = t^2 \left( \frac{2}{t^3} \right) - 2 \left( \frac{1}{t} \right) = \frac{2}{t} - \frac{2}{t} \equiv 0,
\]

which verifies that \( y_2''(t) = 1/t \) is a solution.

Since \( y_2(t) = 1/t \) is discontinuous at \( t = 0 \), \( y_1(t) \) and \( y_2(t) \) are both solutions on either \( -\infty < t < 0 \) or \( 0 < t < \infty \).

(If an interval \( I \) includes \( t = 0 \) then \( y_1, y_2 \) are not both solutions on \( I \), since \( y_2 \) is not continuous on \( I \)).

(b) The Wronskian is

\[
W(y_1, y_2)(t) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = (t^2)(-t^{-2}) - (t^{-1})(2t) = -3
\]

which is nonzero on \( -\infty < t < \infty \). It is also true that the Wronskian is nonzero on any subinterval of the real line, such as \( -\infty < t < 0 \) or \( 0 < t < \infty \).

(c) By the result of part (b) and the theory of linear homogeneous equations, the general solution is

\[
y(t) = c_1 t^2 + c_2 t^{-1},
\]

where \( c_1, c_2 \) are arbitrary constants. The initial conditions require

\[
y(-1) = c_1 - c_2 = -1, \quad y'(-1) = -2c_1 - c_2 = -4,
\]

therefore

\[
c_1 = 1, \quad c_2 = 2,
\]

and the solution to the initial value problem is

\[
y = \phi(t) = t^2 + \frac{2}{t}.
\]
This function is discontinuous at $t = 0$, so $\phi(t)$ is defined on either $-\infty < t < 0$ or $0 < t < \infty$. But the interval has to contain the initial time $t_0 = -1$, so the open interval on which $\phi(t)$ is defined is

$$-\infty < t < 0.$$ 

To use Theorem 3.2.1, we write the IVP as

$$y'' - \frac{2}{t^2} y = 0, \quad y(-1) = -1, \quad y'(-1) = -4.$$ 

Then we see that $p(t) = 0$, $q(t) = -2/t^2$, $g(t) = 0$ are all continuous on the open interval $-\infty < t < 0$ that contains the point $t_0 = -1$, and the theorem implies that $\phi(t)$ exists, is unique, and is defined throughout $-\infty < t < 0$, all of which we have already shown explicitly above (this was guaranteed by Theorem 3.2.1 even before we started to find the solution $y = \phi(t)$ explicitly).

2. (a) $y'' + 3y' + y = 0$

The characteristic equation is

$$r^2 + 3r + 1 = 0,$$

which has the distinct real roots $r_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$, $r_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$, and the general solution is

$$y(t) = c_1 e^{-3 + \sqrt{5} t/2} + c_2 e^{-3 - \sqrt{5} t/2},$$

where $c_1$, $c_2$ are arbitrary constants.

(b) $y'' + \omega^2 y = 0$ ($\omega$ is a given positive constant)

The characteristic equation is

$$r^2 + \omega^2 = 0,$$

which has complex (in fact, “purely imaginary”) roots

$$r_1 = j\omega, \quad r_2 = -j\omega,$$

and (since the real parts are 0) the general solution is

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t,$$

where $c_1$, $c_2$ are arbitrary constants.

(c) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$

The characteristic equation is

$$r^2 + 2r + 2 = 0,$$

which has complex roots

$$r_1 = -1 + j, \quad r_2 = -1 - j,$$

and the general solution is

$$y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t,$$

where $c_1$, $c_2$ are arbitrary constants. Also, we calculate

$$y'(t) = c_1 (-e^{-t} \cos t - e^{-t} \sin t) + c_2 (-e^{-t} \sin t + e^{-t} \cos t).$$
Now we find $c_1$ and $c_2$ to satisfy the IC: evaluate the expressions above at $t = 0$, to get $y(0) = c_1$, $y'(0) = -c_1 + c_2$, therefore we solve
\[ c_1 = 1, \quad -c_1 + c_2 = -1, \]
and get
\[ c_1 = 1, \quad c_2 = 0. \]
The solution to the IVP is
\[ y(t) = e^{-t} \cos t. \]

3. \( t^2y'' + 3ty' + 5y = 0, \ 0 < t < \infty \)
(a) To verify that \( y_1(t) = t^{-1} \cos(2 \ln t) \) is a solution, compute
\[ y_1'(t) = t^{-2}[- \cos(2 \ln t) - 2 \sin(2 \ln t)], \quad y_1''(t) = t^{-3}[-2 \cos(2 \ln t) + 6 \sin(2 \ln t)], \]
then substitute these expressions into the left-hand side of the DE, \( t^2y_1''(t) + 3ty_1'(t) + 5y_1(t) \) and check that the expression simplifies to 0. Similarly for \( y_2(t) = t^{-1} \sin(2 \ln t) \), check with \( y_2'(t) = t^{-2}[2 \cos(2 \ln t) - \sin(2 \ln t)], \quad y_2''(t) = t^{-3}[-6 \cos(2 \ln t) - 2 \sin(2 \ln t)]. \)

Since \( y_1 \) and \( y_2 \) are known to be solutions, to check that they form a fundamental set of solutions, we verify that \( W(y_1, y_2)(t) \) is never 0 on \( 0 < t < \infty \). We calculate
\[
W(y_1, y_2)(t) = \begin{vmatrix}
 t^{-1} \cos(2 \ln t) & t^{-1} \sin(2 \ln t) \\
 t^{-2}[- \cos(2 \ln t) - 2 \sin(2 \ln t)] & t^{-2}[2 \cos(2 \ln t) - \sin(2 \ln t)]
\end{vmatrix} = 2t^{-3}[\cos^2(2 \ln t) + \sin^2(2 \ln t)] = \frac{2}{t^3},
\]
which is never zero on \( 0 < t < \infty \), as required.

(b) \( y(1) = 1, \ y'(1) = 0 \)
The general solution is
\[ y(t) = c_1 t^{-1} \cos(2 \ln t) + c_2 t^{-1} \sin(2 \ln t), \]
and also it is useful to write down (use the results of the Wronskian calculation)
\[ y'(t) = c_1 t^{-2}[ - \cos(2 \ln t) - 2 \sin(2 \ln t)] + c_2 t^{-2}[2 \cos(2 \ln t) - \sin(2 \ln t)]. \]
Evaluating at \( t = 1 \), we have \( y(1) = c_1 \), \( y'(1) = -c_1 + 2c_2 \), and to satisfy the IC we put
\[ c_1 = 1, \quad -c_1 + 2c_2 = 0, \]
therefore
\[ c_1 = 1, \quad c_2 = \frac{1}{2}. \]
The solution to the IVP is
\[ y(t) = t^{-1} \cos(2 \ln t) + \frac{1}{2} t^{-1} \sin(2 \ln t), \quad 0 < t < \infty. \]