1. Find the current $I(t)$ in an $RL$ circuit where $R = 10$ ohms, $L = 2$ henries, the applied voltage is an AC source with $V(t) = \sin(2t)$ and $I(0) = 0$. Hint: you may want to use integration by parts twice.

The ODE is $I' + 5I = (1/2) \sin(2t)$. The integrating factor is $\mu(t) = e^{5t}$.

So, $e^{5t}I = (1/2) \int e^{5t} \sin(2t) dt + C$.

Integrating by parts, we get:

$$\int e^{5t} \sin(2t) dt = -(1/2)e^{5t} \cos(2t) + (5/2) \int e^{5t} \cos(2t) dt$$

Integrating by parts again, we get:

$$\int e^{5t} \cos(2t) dt = (1/2)e^{5t} \sin(2t) - (5/2) \int e^{5t} \sin(2t) dt$$

Putting the previous two equations together, we get:

$$\frac{29}{4} \int e^{5t} \sin(2t) dt = -(1/2)e^{5t} \cos(2t) + (5/4)e^{5t} \sin(2t)$$

and so $\int e^{5t} \sin(2t) dt = -(2/29)e^{5t} \cos(2t) + (5/29)e^{5t} \sin(2t)$. So,

$$I(t) = (-1/29) \cos(2t) + (5/58) \sin(2t) + Ce^{-5t}.$$ 

Using $I(0) = 0$, we get $C = 1/29$ and so

$$I(t) = (-1/29) \cos(2t) + (5/58) \sin(2t) + (1/29)e^{-5t}.$$ 

Note that the solution is the sum of a periodic part, involving sin and cos, and a transient part, $(1/29)e^{-5t}$, which decays to zero as $t \to \infty$. □

2. Consider the initial value problem (IVP): $ty' = y, y(t_0) = y_0$. Find all pairs $(t_0, y_0)$ such that the IVP has

(a) no solution
(b) more than one solution
(c) a unique solution

If $t_0 = 0$, then $y_0 = 0$. So, if $t_0 = 0$ and $y_0 \neq 0$, there is no solution.

If $t_0 \neq 0$, then the ODE is equivalent to the ODE : $y' - (1/t)y = 0$ and the coefficient of $y$ is continuous at $t_0$. So, by Theorem 2.4.1 or 2.4.2 there is a unique solution.

If $t_0 = 0$ and $y_0 = 0$, then $y \equiv 0$ is a solution. But so is $y(t) = ct$ for all $c$ (one can obtain the latter by solving as a variables separable ODE). Note that that the latter contains the former as a special case. □
3. Find all critical points of the following first order autonomous ODE’s and classify them as asymptotically stable, asymptotically unstable, or semi-stable. For each ODE, give rough sketches of the direction field and integral curves.

(a) \( y' = y^4 + y^3 - 2y^2 \)
(b) \( y' = e^y \sin(y) \)

a. The critical points are the zeros of the right hand side, which factors as
\[
y^4 + y^3 - 2y^2 = y^2(y + 2)(y - 1)
\]
So, the critical points are \( y = -2, 0, 1 \).

By testing the sign of this expression in each interval determined by the critical points, we get

<table>
<thead>
<tr>
<th>( y \text{ value} )</th>
<th>( \text{Slope} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y &gt; 1 )</td>
<td>+</td>
</tr>
<tr>
<td>( y = 1 )</td>
<td>0</td>
</tr>
<tr>
<td>( 0 &lt; y &lt; 1 )</td>
<td>-</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>0</td>
</tr>
<tr>
<td>( -2 &lt; y &lt; 0 )</td>
<td>-</td>
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<tr>
<td>( y = -2 )</td>
<td>0</td>
</tr>
<tr>
<td>( y &lt; -2 )</td>
<td>+</td>
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</tbody>
</table>

So, \(-2\) is asymptotically stable, \( 0 \) is asymptotically semi-stable, and \( 1 \) is asymptotically unstable.

b. Since the exponential function is never zero, the critical points are the zeros of \( \sin(y) \), namely \( y = n\pi \) for all integers \( n \).

When \( n \) is even, the sign of \( \sin(y) \) changes from \(-\) to \( +\) as \( y \) increases through \( n\pi \). So, for \( n \) even the critical point is asymptotically unstable.

When \( n \) is odd, the sign of \( \sin(y) \) changes from \( +\) to \(-\) as \( y \) increases through \( n\pi \). So, for \( n \) odd the critical point is asymptotically stable. □

4. Consider the ODE \( 2y'' + 4y' - 6y = 0 \).

(a) Find the general solution of this ODE.
(b) For this ODE, find the solution to the IVP with \( y(0) = 1, y'(0) = 1 \).

a. Divide through by 2, to get \( y'' + 2y' - 3y = 0 \). The characteristic equation is \( r^2 + 2r - 3 = 0 \), which factors as \( (r + 3)(r - 1) = 0 \), whose roots are real and distinct: \(-3, 1\). Thus, the general solution is:
\[
y(t) = c_1 e^{-3t} + c_2 e^t
\]
b. The initial value data gives the linear system:

\begin{align*}
1 &= c_1 + c_2, \\
1 &= -3c_1 + c_2
\end{align*}

The solution to the linear system is \( c_1 = 0, c_2 = 1 \). So, the solution to the IVP is \( y(t) = e^t \). \( \square \)