MATH 256
Homework Assignment 1 Solutions
2015 September 18

1. To verify that the given function is a solution, we differentiate it and substitute it in and see if it satisfies the differential equation, and also set \( t = 0 \) and see if it satisfies the initial condition.

Note that \( e^{t^2} \) and \( \int_0^t e^{-u^2} \, du \) are both continuous functions of \( t \), so \( y(t) \) is continuous (for all \( t, -\infty < t < \infty \)). Differentiating the expression for \( y(t) \), using the product rule and the fundamental theorem of calculus, we get

\[
y'(t) = 2te^{t^2} \int_0^t e^{-u^2} \, du + e^{t^2}e^{-t^2} + 2te^{t^2}
\]

\[
= 2t \left( e^{t^2} \int_0^t e^{-u^2} \, du + e^{t^2} \right) + 1
\]

\[
= 2ty(t) + 1,
\]

for all \( -\infty < t < \infty \), which verifies that the differential equation \( y' - 2ty = 1 \) is satisfied (on the interval \( -\infty < t < \infty \)).

Evaluating the expression at \( t = 0 \) gives

\[
y(0) = e^{0} \cdot 0 + e^{0} = 1,
\]

which verifies that the initial condition is satisfied, and therefore the given expression for \( y(t) \) is indeed a solution of the given initial value problem.

2. (a) \( y' + 2y = te^{-2t}, \quad y(1) = 0 \).

This first order ordinary differential equation is linear. The integrating factor \( \mu(t) \) needs to satisfy \( \mu' = 2\mu \), from which we recognize the solution is \( \mu = \pm e^{2t+k} \), \( k \) arbitrary. We take the + sign and \( k = 0 \) here,

\[
\mu(t) = e^{2t},
\]

and multiply the differential equation by \( \mu(t) \) to get

\[
e^{2t}y' + 2e^{2t}y = t,
\]

\[
(e^{2t}y)' = t,
\]

\[
e^{2t}y = \int t \, dt,
\]

\[
e^{2t}y = \frac{1}{2} t^2 + c,
\]

where \( c \) is an arbitrary constant. Now solve for \( y \): multiply by \( e^{-2t} \) to get the general solution

\[
y(t) = \frac{1}{2} t^2 e^{-2t} + ce^{-2t}
\]

and use the initial condition at \( t = 1 \),

\[
y(1) = \frac{1}{2} e^{-2} + ce^{-2} = 0
\]
to get
\[ c = -\frac{1}{2}, \]
and then the solution to the initial value problem is
\[ y(t) = \frac{1}{2} t^2 e^{-t^2} - \frac{1}{2} e^{-2t}, \]
which is defined in the interval \( -\infty < t < \infty. \)

(b) \( y' + (2/x)y = x^{-2}. \)
This is a linear first order ordinary differential equation. The integrating factor \( \mu(x) \) needs to satisfy \( \mu' = (2/x)\mu, \) so we may take
\[ \mu(x) = e^{\int (2/x) \, dx} \]
\[ = e^{2 \ln x} = x^2 \]
for \( x > 0. \) Multiply the ODE by the integrating factor \( \mu(x) = x^2 \) and get
\[ x^2 y' + 2xy = 1, \]
\[ (x^2 y)' = 1, \]
\[ x^2 y = \int 1 \, dx, \]
\[ x^2 y = x + c, \]
where \( c \) is an arbitrary constant. Divide by \( x^2 \) to get the general solution:
\[ y(x) = \frac{1}{x} + \frac{c}{x^2}, \quad x > 0, \]
where \( c \) is an arbitrary constant.

(c) \( y' = x^2/y(1+x^3). \)
This is a nonlinear first order ordinary differential equation, and it is separable. Write the ODE as
\[ \frac{dy}{dx} = \frac{x^2}{y(1+x^3)}, \]
and separate variables to get
\[ y \, dy = \frac{x^2}{1+x^3} \, dx. \]
Integrate both sides to get
\[ \frac{1}{2} y^2 = \frac{1}{3} \ln |1 + x^3| + c, \]
and solve for \( y \) to get
\[ y(x) = \pm \sqrt{\frac{2}{3} \ln |1 + x^3| + c}, \]
where \( c \) is an arbitrary constant.
(d) \( y' = (1 - 2x)y^2, \ y(0) = -1/6. \)  
This first order ordinary differential equation is nonlinear and separable. Write the ODE as  
\[-y^{-2} \, dy = (2x - 1) \, dx,\]  
then integrate both sides to obtain  
\[y^{-1} = x^2 - x + c,\]  
where \( c \) is an arbitrary constant. Substituting \( x = 0 \) and \( y = -1/6 \), we find that  
\[c = -6,\]  
then \( y^{-1} = x^2 - x - 6 \), or putting the solution into explicit form  
\[y(x) = \frac{1}{x^2 - x - 6}.\]  
Noting that \( x^2 - x - 6 = (x + 2)(x - 3) \), we see that for \( y(x) \) to be defined, we need the denominator to be nonzero, so \( x \neq -2 \) and \( x \neq 3 \). The interval in which the solution is defined must contain the point \( x = 0 \) where the initial condition is given, so the interval in which the solution is defined is  
\[-2 < x < 3.\]  
3. Letting \( y = e^{rt} \), we calculate \( y' = re^{rt} \) and substituting into the differential equation gives  
\[a \left( re^{rt} \right) + b \left( e^{rt} \right) = 0.\]  
Dividing by \( e^{rt} \) (which is never zero) gives the algebraic equation  
\[ar + b = 0.\]  
This is easily solved to give  
\[r = -\frac{b}{a}.\]  
Now verify that \( y(t) = e^{-(b/a)t} \) is indeed a solution, by substituting it into the differential equation:  
\[a \left( -\frac{b}{a} e^{-bt/a} \right) + b \left( e^{-bt/a} \right) = -be^{-bt/a} + be^{bt/a} = 0,\]  
for all \( t \) (i.e., \(-\infty < t < \infty\)).  
4. The current \( i(t) \) in the \( LR \) series circuit satifies the initial value problem  
\[20 \, i' + 2 \, i = v(t), \quad i(0) = 0,\]  
or  
\[i' + \frac{1}{10} \, i = \frac{1}{20} \, v(t), \quad i(0) = 0,\]  
where \( v(t) \) is as given. Since \( v(t) \) is defined piecewise, on two different intervals, we solve separately for \( i(t) \) on the different intervals.
On the interval $0 \leq t \leq 20$ we have

$$i' + \frac{1}{10} i = 6, \quad i(0) = 0.$$  

This is a linear first order ODE which we can solve by multiplying by the integrating factor

$$\mu(t) = e^{\int(1/10)\,dt} = e^{t/10},$$

to obtain

$$e^{t/10} i' + \frac{1}{10} e^{t/10} i = 6 e^{t/10},$$

$$(e^{t/10} i)' = 6 e^{t/10},$$

$$e^{t/10} i = 6 \int e^{t/10} \, dt,$$

$$e^{t/10} i = 60 e^{t/10} + c_1,$$

where $c_1$ is a constant. Substituting $t = 0$ and $i = 0$ at the IC, we get

$$c_1 = -60$$

and multiplying by $e^{-t/10}$ we have the current on the first interval,

$$i(t) = 60 - 60 e^{-t/10}, \quad 0 \leq t \leq 20.$$  

Now on the second interval $t > 20$ we have

$$i' + \frac{1}{10} i = 0,$$

which is easily solved (can use integrating factor method with the same integrating factor, but can also use first-year calculus methods for exponential growth and decay) as

$$i(t) = c_2 e^{-t/10}, \quad t > 20,$$

where $c_2$ is a constant.

To find $c_2$, we use the hint and choose $c_2$ so that $i(t)$ is continuous everywhere, in particular at $t = 20$: we require

$$\lim_{t \to 20^-} i(t) = \lim_{t \to 20^+} i(t).$$

Approaching $t = 20$ from below we have

$$\lim_{t \to 20^-} i(t) = \lim_{t \to 20^-} (60 - 60 e^{-t/10}) = 60 - 60 e^{-2} = i(20).$$

while approaching $t = 20$ from above we have

$$\lim_{t \to 20^+} i(t) = \lim_{t \to 20^+} c_2 e^{-t/10} = c_2 e^{-2}.$$
Therefore in order for \( i(t) \) to be continuous we must have
\[
60 - 60 e^{-t} = c_2 e^{-t},
\]
which gives
\[
c_2 = 60 e^2 - 60.
\]
The current \( i(t) \) is therefore given by
\[
i(t) = \begin{cases} 
60 - 60 e^{-t/10} & \text{if } 0 \leq t \leq 20, \\
(60 e^2 - 60) e^{-t/10} & \text{if } t > 20.
\end{cases}
\]

5. We need to solve the IVP for \( q(t) \),
\[
(k_1 + k_2 t) q' + \frac{1}{C} q = V, \quad q(0) = q_0.
\]
The ODE is linear, first order. But before using the integrating factor method, we should first divide by the coefficient \( k_1 + k_2 t \) of \( q' \) in the differential equation to get
\[
q' + \frac{1}{C (k_1 + k_2 t)} q = \frac{V}{k_1 + k_2 t}.
\]
Now we can find the integrating factor as
\[
\mu(t) = e^{\int \frac{1}{(C(k_1 + k_2 t))} dt} = e^{(1/Ck_2) \ln(k_1 + k_2 t)} = (k_1 + k_2 t)^{1/Ck_2},
\]
where we are assuming that the resistance \( k_1 + k_2 t \) is always positive, so \(|k_1 + k_2 t| = k_1 + k_2 t\). Multiplying the latter form of the differential equation by the integrating factor, then integrating, we get
\[
(k_1 + k_2 t)^{1/Ck_2} q' + \frac{1}{C} (k_1 + k_2 t)^{1/(Ck_2)} - 1 q = V (k_1 + k_2 t)^{1/(Ck_2)} - 1,
\]
\[
\left[(k_1 + k_2 t)^{1/(Ck_2)} q\right]' = V (k_1 + k_2 t)^{1/(Ck_2)} - 1,
\]
\[
(k_1 + k_2 t)^{1/(Ck_2)} q = V \int (k_1 + k_2 t)^{1/(Ck_2)} - 1 dt,
\]
\[
(k_1 + k_2 t)^{1/(Ck_2)} q = VC (k_1 + k_2 t)^{1/(Ck_2)} + b,
\]
where \( b \) is a constant (not using \( c \) for the integration constant here so it doesn’t get confused with the capacitance \( C \)). Solving for \( q \) we get
\[
q = VC + b(k_1 + k_2 t)^{-1/(Ck_2)}.
\]
Substuting \( t = 0 \) and \( q = q_0 \) at the initial condition, we have
\[
q_0 = VC + bk_1^{-1/(Ck_2)}
\]
and thus

\[ b = (q_0 - VC) \, k_1^{1/(Ck_2)}. \]

Using this value for \( b \) we get the charge on the capacitor

\[
q(t) = VC + (q_0 - VC) \, k_1^{1/(Ck_2)} \, (k_1 + k_2t)^{-1/(Ck_2)} \\
= VC + (q_0 - VC) \left( \frac{k_1}{k_1 + k_2t} \right)^{1/(Ck_2)}
\]