MATH 256  
Homework Assignment 10 Solutions  
November 2015

1. (a) Substituting $u(x,t) = X(x)T(t)$ into the PDE, we get

$$X(x)\dot{T}(t) = \alpha^2 X''(x)T(t) \quad \text{for all } 0 < x < \pi \text{ and } t > 0,$$

where $\dot{} = d/dt$ and $' = d/dx$. Dividing by $\alpha^2 X(x)T(t)$ gives

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} \quad \text{for all } x \text{ and } t.$$

The left hand side is a function only of $t$, independent of $x$, while the right hand side is a function only of $x$, independent of $t$. The only way equality can hold for all $x$ and $t$ is for both sides to equal a constant, which we call $-\lambda$. Thus $X(x)$ and $T(t)$ are required to satisfy the ordinary differential equations

$$X''(x) + \lambda X(x) = 0, \quad \dot{T}(t) + \alpha^2 \lambda T(t) = 0.$$

(b) If $u(x,t) = X(x)T(t)$, then the boundary condition $u_x(0,t) = X'(0)T(t) = 0$ for all $t > 0$ is satisfied if

$$X'(0) = 0.$$

Similarly, the boundary condition $u(\pi,t) = X(\pi)T(t) = 0$ for all $t > 0$ is satisfied if

$$X(\pi) = 0.$$

So we need to find the eigenvalues and eigenfunctions of the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi, \quad X'(0) = 0, \quad X(\pi) = 0.$$

i) Suppose $\lambda < 0$. Then $-\lambda > 0$. The roots of the characteristic equation $r^2 + \lambda = 0$ are $r_1 = \sqrt{-\lambda}$, $r_2 = -\sqrt{-\lambda}$ which are real and distinct if $\lambda < 0$, and the general solution of the ODE is

$$X(x) = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x},$$

where $c_1$ and $c_2$ are arbitrary constants. The derivative is

$$X'(x) = c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda} x} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda} x}.$$

At the boundary $x = 0$ we must have

$$X'(0) = c_1 \sqrt{-\lambda} - c_2 \sqrt{-\lambda} = 0,$$

where $\sqrt{-\lambda} > 0$, so $c_2 = c_1$ and

$$X(x) = c_1 \left( e^{\sqrt{-\lambda} x} + e^{-\sqrt{-\lambda} x} \right).$$
Evaluating at the boundary $x = \pi$ we must have

$$X(\pi) = c_1 \left( e^{\sqrt{(-\lambda)\pi}} + e^{-\sqrt{(-\lambda)\pi}} \right) = 0.$$ 

Both terms in the parentheses are positive, so we are forced to take

$$c_1 = 0,$$

then $X(x) \equiv 0$. So there are no nontrivial solutions $X(x)$ if $\lambda < 0$, i.e. $\lambda < 0$ cannot be an eigenvalue.

ii) Suppose $\lambda = 0$. Then the ODE for $X(x)$ becomes

$$X'' = 0$$

(the roots of the characteristic equation are $r_1 = 0$, $r_2 = 0$) and the general solution is

$$X(x) = c_1 + c_2 x,$$

where $c_1$ and $c_2$ are arbitrary constants. At the boundary $x = 0$ we require

$$X'(0) = c_2 = 0,$$

so

$$X(x) = c_1.$$

At the boundary $x = \pi$ we require

$$X(\pi) = c_1 = 0,$$

so $X(x) \equiv 0$ and there are no nontrivial solutions $X(x)$ if $\lambda = 0$, i.e. $\lambda = 0$ cannot be an eigenvalue.

iii) Suppose $\lambda > 0$. Then (the roots of the characteristic equation $r_1 = j\sqrt{\lambda}$, $r_2 = -j\sqrt{\lambda}$ are complex) the general solution of the ODE for $X(x)$ is in this case

$$X(x) = c_1 \cos \left( \sqrt{\lambda} x \right) + c_2 \sin \left( \sqrt{\lambda} x \right),$$

where $c_1$ and $c_2$ are arbitrary constants. At the boundary $x = 0$ we require

$$X'(0) = -c_1 \sqrt{\lambda} \sin(0) + c_2 \sqrt{\lambda} \cos(0) = c_2 \sqrt{\lambda} = 0,$$

so $c_2 = 0$ (since $\sqrt{\lambda} > 0$) and

$$X(x) = c_1 \cos \left( \sqrt{\lambda} x \right).$$

At the boundary $x = \pi$ we require

$$X(\pi) = c_1 \cos \left( \sqrt{\lambda} \pi \right) = 0.$$
A solution with \( c_1 \neq 0 \) is possible, if and only if
\[
\cos \left( \sqrt{\lambda} \pi \right) = 0,
\]
and all possible \( \sqrt{\lambda} \) are \( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \), or \( \sqrt{\lambda} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \), or
\[
\sqrt{\lambda} = \frac{2n-1}{2}, \quad n = 1, 2, 3, \ldots.
\]
Therefore all the possible eigenvalues (recall we showed none are negative or zero) are
\[
\lambda_n = \frac{(2n-1)^2}{4}, \quad n = 1, 2, 3, \ldots,
\]
and the corresponding eigenfunctions are
\[
X_n(x) = \cos \left( \frac{(2n-1)x}{2} \right)
\]
(times any nonzero constant).

(c) When \( \lambda = \lambda_n \), the corresponding functions \( T(t) = T_n(t) \) must satisfy the ODE
\[
\dot{T} + \alpha^2 \lambda_n T = 0,
\]
and the general solution is
\[
T_n(t) = ce^{-\alpha^2 \lambda_n t} = ce^{-\alpha^2(2n-1)^2t/4}
\]
where \( c \) is an arbitrary constant (a different one for each \( n \)). We have infinitely many solutions of the PDE and BC,
\[
u_n(x, t) = X_n(x)T_n(t) = e^{-\alpha^2(2n-1)^2t/4} \cos \left( \frac{(2n-1)x}{2} \right)
\]
and therefore any linear superposition (i.e. series) of such solutions
\[
u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2(2n-1)^2t/4} \cos \left( \frac{(2n-1)x}{2} \right)
\]
also satisfies (at least formally, i.e. without worrying about theorems on convergence) the PDE and BC. Finally, we can satisfy the initial condition at \( t = 0 \)
\[
u(x, 0) = \sum_{n=1}^{\infty} c_n \cos \left( \frac{(2n-1)x}{2} \right) = f(x),
\]
by taking \( c_n \) to be Fourier cosine series coefficients
\[
c_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos \left( \frac{(2n-1)x}{2} \right) \, dx, \quad n = 1, 2, 3, \ldots.
\]
(see Suggested Homework Problem 10.4.40 for a discussion of convergence of cosine series with only odd-numbered spatial frequencies).
2. (a) Substituting $u(x,t) = X(x)T(t)$ into the PDE $u_{tt} + u_t = u_{xx}$, we get

$$X(x)\ddot{T}(t) + X(x)\dot{T}(t) = X''(x)T(t) \quad \text{for all } 0 < x < 2 \text{ and } t > 0.$$ 

Dividing by $X(x)T(t)$ gives

$$\frac{\ddot{T}(t) + \dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} \quad \text{for all } 0 < x < 2 \text{ and } t > 0.$$ 

The left hand side is a function only of $t$, independent of $x$, while the right hand side is only a function of $x$, independent of $t$. The only way equality can hold for all $x$ and $t$ is for both sides to equal a constant, which we call $-\lambda$. Thus $X(x)$ and $T(t)$ are required to satisfy the ODEs

$$X'' + \lambda X = 0, \quad \ddot{T} + \dot{T} + \lambda T = 0.$$ 

(b) If $u(x,t) = X(x)T(t)$, then the boundary conditions $u(0,t) = X(0)T(t) = 0$ and $u(2,t) = X(2)T(t) = 0$ for all $t > 0$ are satisfied if $X(0) = 0$ and $X(2) = 0$. We need the eigenvalues and eigenfunctions of the boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(2) = 0.$$ 

This was done in the lectures (use $\ell = 2$ here), so do not have to be re-derived. The eigenvalues are known to be

$$\lambda_n = \frac{n^2 \pi^2}{4}, \quad n = 1, 2, 3, \ldots$$

and the corresponding eigenfunctions are known to be

$$X_n(x) = \sin \frac{n\pi x}{2}$$

(times an arbitrary nonzero constant).

(c) When $\lambda = \lambda_n$, the corresponding functions $T = T_n(t)$ must satisfy the second-order linear ODE

$$\ddot{T} + \dot{T} + \lambda_n T = 0,$$

whose characteristic equation

$$r^2 + r + \lambda_n = 0$$

has roots

$$r_{1,2} = (-1 \pm \sqrt{1 - 4\lambda_n})/2 = (-1 \pm \sqrt{1 - n^2 \pi^2})/2 = -\frac{1}{2} \pm j\frac{\sqrt{n^2 \pi^2 - 1}}{2}$$

(note that $n^2 \pi^2 - 1 > 0$) for all $n = 1, 2, 3, \ldots$. The general solution is

$$T_n(t) = c_n e^{-t/2} \cos\left(\frac{\sqrt{n^2 \pi^2 - 1}}{2} t\right) + d_n e^{-t/2} \sin\left(\frac{\sqrt{n^2 \pi^2 - 1}}{2} t\right)$$

where $c_n$ and $d_n$ are arbitrary constants for each $n$. We have infinitely many solutions of the PDE and BC,

$$u_n(x,t) = X_n(x)T_n(t) = \left[ c_n e^{-t/2} \cos\left(\frac{\sqrt{n^2 \pi^2 - 1}}{2} t\right) + d_n e^{-t/2} \sin\left(\frac{\sqrt{n^2 \pi^2 - 1}}{2} t\right) \right] \sin\left(\frac{n\pi x}{2}\right).$$

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and therefore any “sum” (i.e. series) of solutions
\[
u(x,t) = \sum_{n=1}^{\infty} \left[ c_n e^{-t/2} \cos \left( \frac{\sqrt{n^2 \pi^2 - 1}}{2} t \right) + d_n e^{-t/2} \sin \left( \frac{\sqrt{n^2 \pi^2 - 1}}{2} t \right) \right] \sin \left( \frac{n \pi x}{\ell} \right)
\]
is also a solution of the PDE and BC. We satisfy the initial condition at \( t = 0 \)
\[
u(x,0) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{\ell} \right) = c_1 \sin \left( \frac{\pi x}{2} \right) + c_2 \sin \left( \frac{2 \pi x}{2} \right) + c_3 \sin \left( \frac{3 \pi x}{2} \right) + \cdots = \sin \left( \frac{\pi x}{2} \right)
\]
by taking
\[
\begin{align*}
c_1 &= 1, \\
c_2 &= 0, \\
c_3 &= 0, \\
&\vdots
\end{align*}
\]
i.e. \( c_n = 0 \) for all \( n \geq 2 \) (this can also be done by using the formula for Fourier sine series and trig identities but that calculation is much longer). Here the initial condition is seen to already be expressed as a Fourier sine series of an especially simple kind, as seen in HW 9: one coefficient is nonzero, the rest are all zero. Taking the partial derivative of the series for \( u(x,t) \) with respect to \( t \) and then setting \( t = 0 \), we satisfy the other initial condition
\[
u_t(x,0) = \sum_{n=1}^{\infty} \left[ -\frac{1}{2} c_n + \frac{\sqrt{n^2 \pi^2 - 1}}{2} d_n \right] \sin \left( \frac{n \pi x}{\ell} \right) = -\sin(2\pi x)
\]
by recognizing again that the right hand side is already a trivial sort of Fourier sine series with all coefficients but one (\( n = 4 \)) equal to zero. Matching coefficients, we have
\[
-\frac{1}{2} c_4 + \frac{\sqrt{16 \pi^2 - 1}}{2} d_4 = -1, \quad \left( -\frac{1}{2} c_n + \frac{\sqrt{n^2 \pi^2 - 1}}{2} d_n \right) = 0
\]
for all \( n \neq 4 \). Since \( c_1 = 1 \) and \( c_2 = 0 \) for \( n \neq 1 \) we can solve for all \( d_n \),
\[
d_1 = \frac{1}{\sqrt{\pi^2 - 1}}, \quad d_4 = \frac{2}{\sqrt{16 \pi^2 - 1}}, \quad d_n = 0 \quad \text{for} \quad n = 2, 3 \quad \text{and} \quad n = 5, 6, 7, \ldots.
\]
Putting everything together, we have finally
\[
u(x,t) = \left[ e^{-t/2} \cos \left( \frac{\sqrt{\pi^2 - 1}}{2} t \right) + \frac{1}{\sqrt{\pi^2 - 1}} e^{-t/2} \sin \left( \frac{\sqrt{\pi^2 - 1}}{2} t \right) \right] \sin \left( \frac{\pi x}{2} \right)
\]
\[
-\frac{2}{\sqrt{16 \pi^2 - 1}} e^{-t/2} \sin \left( \frac{\sqrt{16 \pi^2 - 1}}{2} t \right) \sin(2\pi x).
\]

3. As seen in the lectures, the solution of the wave equation with homogeneous boundary conditions is
\[
u(x,t) = \sum_{n=1}^{\infty} \left( c_n \cos \frac{n \pi a t}{\ell} + d_n \sin \frac{n \pi a t}{\ell} \right) \sin \frac{n \pi x}{\ell}
\]
where
\[
c_n = \frac{2}{\ell} \int_0^\ell f(x) \sin \frac{n \pi x}{\ell} \, dx, \quad d_n = \frac{2}{n \pi a} \int_0^\ell g(x) \sin \frac{n \pi x}{\ell} \, dx.
\]
Here we have \( a = 5, \ell = 4, f(x) \equiv 0 \), so
\[
c_n = 0 \quad \text{for all} \quad n = 1, 2, 3, \ldots,
\]
and

\[
d_n = \frac{2}{n\pi} \int_0^4 g(x) \sin \frac{n\pi x}{4} \, dx \\
= \frac{2}{5\pi} \int_1^3 3 \sin \frac{n\pi x}{4} \, dx \\
= \frac{2}{5\pi} \left( -\frac{12}{\pi} \right) \cos \frac{n\pi x}{4} \bigg|_1^3 \\
= \frac{24}{5\pi^2 n^2} \left( \cos \frac{n\pi}{4} - \cos \frac{3n\pi}{4} \right)
\]

and the solution of the initial boundary value problem is

\[
u(x, t) = \frac{24}{5\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi/4) - \cos(3n\pi/4)}{n^2} \sin \frac{n\pi x}{4} \sin \frac{n\pi x}{4}.
\]