JUSTIFY ALL OF YOUR ANSWERS. YOU MAY USE RESULTS FROM
CLASS AND HOMEWORK.

CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.
THERE ARE 4 PROBLEMS ON THIS EXAM.
1. (25 marks)
Let $\mathbf{F} = (F_1, F_2, F_3)$ be a $C^2$ vector field on $\mathbb{R}^3$. Show that
\[ \nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla F_1, \nabla \cdot \nabla F_2, \nabla \cdot \nabla F_3) \]
by showing that the left and right hand sides of the equation agree in the first component and stating in one sentence why this suffices to prove the equation.
Solution: a.

\[
\nabla \times \vec{F} = \left| \begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{array} \right| = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

Thus, the first component of the left hand side of the equation to be proved is

\[
\frac{\partial}{\partial y} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \frac{\partial}{\partial z} \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) = \frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial x} \quad (1)
\]

The first component of \( \nabla(\nabla \cdot \vec{F}) \) is

\[
\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} \quad (2)
\]

And \( \nabla \cdot \nabla F_1 \) is

\[
\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \quad (3)
\]

Equation (1) = Equation (2) - Equation (3) because of equality of mixed partials of \( C^2 \) functions.

Equality in the second and third components is obtained from the above argument by symmetry.
2. (25 marks)

Let $C$ be the unit circle with \textit{clockwise} orientation. Let

$$F(x, y) = (\sqrt{2 + 3x^2 + y^3}, e^{y^2} + 6xy^2)$$

Find $\int_C F \cdot d\overline{r}$. 
By Green’s theorem,

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = - \int \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA = - \int \int_D 3y^2 \, dxdy \]

where \( D \) is the unit disk (the negative sign in front accounts for the clockwise orientation). Using polar coordinates, this equals

\[ -3 \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^3 \sin^2(\theta) \, drd\theta = -3 \left( \int_{\theta=0}^{2\pi} \sin^2(\theta) \, d\theta \right) \left( \int_{r=0}^{1} r^3 \, dr \right) \]

\[ = -3/8 \left( \int_{\theta=0}^{2\pi} (1 - \cos(2\theta)) \, d\theta \right) = -(3/4)\pi. \]
3.

(25 marks)
Let \((x_i, y_i), i = 1, \ldots, n\) be distinct points in the plane. Define \((x_{n+1}, y_{n+1}) := (x_1, y_1)\). Assume that the line segments from \((x_i, y_i)\) to \((x_{i+1}, y_{i+1})\), \(i = 1, \ldots, n\) do not intersect (except of course that the line segment from \((x_i, y_i)\) to \((x_{i+1}, y_{i+1})\) intersects the line segment from \((x_{i+1}, y_{i+1})\) to \((x_{i+2}, y_{i+2})\) at \((x_{i+1}, y_{i+1})\)). So, these line segments form the boundary of a polygon. Using vector calculus, find the area of the polygon. Your answer should be given in terms of the numbers \(x_1, \ldots, x_n, y_1, \ldots, y_n\).
Solution: By Green’s theorem, the area of the polygon is the same as the line integral of the vector field \( \mathbf{F}(x, y) = (0, x) \) around the boundary \( C \) of the polygon, provided that \( C \) is parameterized with positive orientation. Below, we will parameterize \( C \) in a way that may or may not be positively oriented. So, we will take the absolute value of the result.

Parameterize the edge from \((x_i, y_i)\) to \((x_{i+1}, y_{i+1})\) by \( \mathbf{r}(t) = t((x_{i+1}, y_{i+1})) + (1 - t)(x_i, y_i) \), \( 0 \leq t \leq 1 \). Then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \int_0^1 \left(0, tx_{i+1} + (1 - t)x_i\right) \cdot \left(x_{i+1} - x_i, y_{i+1} - y_i\right) dt
\]

where \( x_{n+1} = x_1 \) and \( y_{n+1} = y_1 \). This equals

\[
\sum_{i=1}^{n} \int_0^1 (tx_{i+1} + (1 - t)x_i)(y_{i+1} - y_i) \, dt
\]

\[
= \sum_{i=1}^{n} (y_{i+1} - y_i)((1/2)x_{i+1} - (1/2)x_i + x_i)
\]

\[
= (1/2) \sum_{i=1}^{n} (y_{i+1} - y_i)(x_{i+1} + x_i).
\]

So the area is

\[
(1/2)|\sum_{i=1}^{n} (y_{i+1} - y_i)(x_{i+1} + x_i)|.
\]
4. Let
\[ D = \{(u, v) : 0 \leq u \leq 2\pi, \ -1 \leq v \leq 1\} \]
and \( \bar{r} : D \rightarrow \mathbb{R}^3 \) be defined by:
\[ \bar{r}(u, v) = \left( (1 + \frac{1}{2}v \cos(u)) \cos(u), \ (1 + \frac{1}{2}v \cos(u)) \sin(u), \ \frac{1}{2}v \sin(u) \right) \]

(a) (18 marks) Show that \( \bar{r} \) is one-to-one on the interior of \( D \) (i.e, on the set \( \{(u, v) : 0 < u < 2\pi, \ -1 < v < 1\}\)).

(b) (7 marks) Find all values of \( u, v, u', v' \) such that \( \bar{r}(u, v) = \bar{r}(u', v') \).
Solution:

a. Let \((u, v) \in D\) and \((x, y, z) = \tilde{r}(u, v)\). Since \(1 + \frac{1}{2}v\cos\left(\frac{u}{2}\right) \geq 1/2 > 0\), \(x\) and \(\cos(u)\) have the same sign. Similarly, \(y\) and \(\sin(u)\) have the same sign. Since \(\tan(u) = y/x\), we can uniquely determine \(u\) up to \(\pm 2\pi\) from \(x\) and \(y\). From \(z\), we can then determine \(v\), unless \(u = 0\) or \(2\pi\). This implies that \(\tilde{r}\) is one-to-one on the interior of \(D\).

b. By the argument in part (a), if \(\tilde{r}((u, v)) = \tilde{r}((u', v'))\) and \((u, v) \neq (u', v')\), then either (1) \(\{u, u'\} = \{0, 2\pi\}\) or (2) \(u = u'\).

Case 1: WLOG \(u = 0\) and \(u' = 2\pi\). Then \(u/2 = 0\) and \(u'/2 = \pi\). Then, \(1 + (1/2)v = x = 1 - (1/2)v'\), and so \(v' = -v\). Thus, the only possibility is:

\[
  u = 0, u' = 2\pi, v = -v' \quad (4)
\]

Case 2: We can determine \(v\) from \(z\) unless \(u = u' = 0\) or \(u = u' = 2\pi\). If \(u = u' = 0\), then \(1 + (1/2)v = x = 1 + (1/2)v'\) and so \(v = v'\). If \(u = u' = 2\pi\), then \(1 - (1/2)v = x = 1 - (1/2)v'\) and so again \(v = v'\).

Thus, the only possibility is (4) above.

This is the Mobius band (see how the sides of \(D\) are glued by the map \(\tilde{r}\)).