Lecture 9:
Recall: *Frenet Formula*

' denotes \( d/ds \).

**Fundamental Theorem of Space Curves:** Let \( C_1 \) and \( C_2 \) be very smooth oriented curves in \( \mathbb{R}^3 \), with nonzero curvature. Then \( C_1 \) and \( C_2 \) are congruent iff

1. for all \( s \), \( \kappa_1(s) = \kappa_2(s) \) —and—

2. either A) for all \( s \), \( \tau_1(s) = \tau_2(s) \)

   —or—

   B) for all \( s \), \( \tau_1(s) = -\tau_2(s) \)

Last time, we started to prove the “Only If.” We showed that if \( \overline{r}_1(s), \overline{r}_2(s) \) denote the arclength parameterizations of \( C_1 \) and \( C_2 \) and \( R \) is a rigid motion that maps \( C_1 \) to \( C_2 \), then \( R(\overline{r}_1(s)) = \overline{r}_2(s) \) for all \( s \).

The proof showed that if \( L \) is any linear transformation and \( \overline{g}(t) \) is a differentiable vector function, then \( L(\frac{d\overline{g}(t)}{dt}) = \frac{d(L\overline{g})}{dt} \).

Continuing with the proof of “Only If:”

Case 1: \( R \) is a translation. HW4: show that in this case in fact \( \kappa_1(s) = \kappa_2(s) \) and \( \tau_1(s) = \tau_2(s) \).

Case 2: \( R \) is a rigid motion, which is a composition only of rotations and reflections (in particular is a linear transformation).

We will show that \( R \) maps the moving from on \( C_1 \) to the moving frame on \( C_2 \).

Since \( R \) is a linear transformation, we have

\[
T_2(s) = \overline{r}_2'(s) = (R(\overline{r}_1(s)))' = R(\overline{r}_1'(s)) = R(T_1(s)) \quad (1)
\]
Thus,
\[ T'_2(s) = (R(T_1(s)))' = R(T'_1(s)) \] (2)

Thus, since an isometry preserves lengths of vectors,
\[ \kappa_2(s) = |T'_2(s)| = |R(T'_1(s))| = |T'_1(s)| = \kappa_1(s) \]

And normalizing (2) by \( \kappa(s) \), we get \( N_2(s) = R(N_1(s)) \).

Since \( R \) preserves angles, it must map \( B_1(s) \) to \( \pm B_2(s) \).

Case A: \( R(B_1(s)) = B_2(s) \)

Case B: \( R(B_1(s)) = -B_2(s) \) (reflection)

Proof for Case A: By Frenet, for \( i = 1, 2 \),
\[ \tau_i(s) = B_i(s) \cdot N'_i(s) \]

Again since \( R \) is a linear transformation and \( N_2(s) = R(N_1(s)) \),
we have \( N'_2(s) = R(N'_1(s)) \).

Since \( R \) preserves dot products,
\[ \tau_2(s) = B_2(s) \cdot N'_2(s) = R(B_1(s)) \cdot R(N'_1(s)) = (B_1(s)) \cdot N'_1(s) = \tau_1(s). \]

For Case B, apply all the steps of the proof of Case A, and see that you get a minus sign in the last equation.

End of “Only if.”

Tacitly, we are assuming that if \( C_1 \) and \( C_2 \) have finite length, then they have the same length.

\textbf{If:}

Let \( \bar{r}_1, \bar{r}_2 \) be arclength parametrizations for \( C_1, C_2 \).

We assume \([0, 1]\) is the common domain for \( \bar{r}_1, \bar{r}_2 \).

Note that translations and rotations preserve the frame, curvature and torsion at all \( s \in [0, 1] \). (by the ~ \( \text{iff} \) ~)

\[ \text{by the } \text{iff} \]

\[ \text{by the } \text{iff} \]
Step 1: Translate $\mathbf{r}_2$ so that $\mathbf{r}_1(0) = \mathbf{r}_2(0)$.

Step 2:

Proposition: There exists a rotation around the point $\mathbf{r}_1(0) = \mathbf{r}_2(0)$ so that $\mathbf{T}_1(0) = \mathbf{T}_2(0), \mathbf{N}_1(0) = \mathbf{N}_2(0), \mathbf{B}_1(0) = \mathbf{B}_2(0)$.

Proof:

If $\mathbf{T}_1(0) \neq \mathbf{T}_2(0)$, rotate $C_2$ in the plane spanned by $\mathbf{T}_1(0), \mathbf{T}_2(0)$ to make $\mathbf{T}_1(0) = \mathbf{T}_2(0)$ (if $\mathbf{T}_1(0) = -\mathbf{T}_2(0)$, rotate by 180 degrees in any plane containing $\mathbf{T}_1(0)$). This preserves curvature and torsion.

Note that $\mathbf{N}_1(0), \mathbf{N}_2(0)$ are in a plane through $\mathbf{r}_1(0)$ perpendicular to $\mathbf{T}_1(0) = \mathbf{T}_2(0)$; if $\mathbf{N}_1(0) \neq \mathbf{N}_2(0)$, rotate $C_2$ in that plane so that $\mathbf{N}_1(0)$ becomes $\mathbf{N}_2(0)$; note that we still have $\mathbf{T}_1(0) = \mathbf{T}_2(0)$. This still preserves curvature and torsion.

Finally, $\mathbf{B}_1(0) = \mathbf{T}_1(0) \times \mathbf{N}_1(0) = \mathbf{T}_2(0) \times \mathbf{N}_2(0) = \mathbf{B}_2(0)$.

So, at this point we may assume that the curves $C_1$ and $C_2$ have the same starting point $\mathbf{r}_1(0) = \mathbf{r}_2(0)$ and the same $T, N, B$ frame at that point.

Step 3:

Key Proposition: Let $\mathbf{r}_1, \mathbf{r}_2 : [0, 1] \to \mathbb{R}^3$ be very smooth arclength parametrizations of oriented curves, with nonzero curvature, such that

1. $\mathbf{r}_1(0) = \mathbf{r}_2(0), \mathbf{T}_1(0) = \mathbf{T}_2(0), \mathbf{N}_1(0) = \mathbf{N}_2(0), \mathbf{B}_1(0) = \mathbf{B}_2(0)$

2. for all $s$, $\kappa_1(s) = \kappa_2(s)$

3. for all $s$, $\tau_1(s) = \tau_2(s)$

Then for all $s$, $\mathbf{r}_1(s) = \mathbf{r}_2(s)$.

Proof:
For $i = 1, 2$, let

$$X_i(s) = \begin{bmatrix} T_i(s) \\ N_i(s) \\ B_i(s) \end{bmatrix}$$

Frenet formula can be recast as:

$$X'_i = AX_i, \quad i = 1, 2$$

(can view $A(s)$ as a $9 \times 9$ matrix).

We will show that $X_1(s) = X_2(s)$ for all $s$, but all we need from this is: $T_1(s) = T_2(s)$ for all $s$; for then $\bar{\tau}_1(s)$ and $\bar{\tau}_2(s)$ are antiderivatives of the same vector valued function with the same initial condition, $\bar{\tau}_1(0) = \bar{\tau}_2(0)$, and thus $\bar{\tau}_1(s)$ and $\bar{\tau}_2(s)$ are identical for all $s$. (Arnold's note: we can assume $\kappa = \kappa$ and $\iota = \iota$)

Let

$$f(s) := T_1(s) \cdot T_2(s) + N_1(s) \cdot N_2(s) + B_1(s) \cdot B_2(s)$$

Observe

$$f(s) \leq 3, \quad f(0) = 3.$$ 

and for any $s$, $f(s) = 3$ iff $X_1(s) = X_2(s)$.

Will show:

$$f'(s) = 0.$$ 

for all $s$; for then $f(s) = 3$ and so $X_1(s) = X_2(s)$, as desired.

Proof of $f'(s) = 0$ for all $s$. (Eric Naslund):

Let $\kappa(s), \tau(s)$ be the common value of curvature and torsion for the two curves. Let

$$A(s) := \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}$$
Lecture 10:
Recall Frenet Formulas, Fundamental Theorem of Space Curves (Theorem 3, p. 650), Key Proposition:
Proof of Key Proposition:
Let
\[ f(s) := \mathbf{T}_1(s) \cdot \mathbf{T}_2(s) + \mathbf{N}_1(s) \cdot \mathbf{N}_2(s) + \mathbf{B}_1(s) \cdot \mathbf{B}_2(s) \]
Observe
\[ f(s) \leq 3, \quad f(0) = 3. \]
and for any \( s \), \( f(s) = 3 \) iff \( \mathbf{X}_1(s) = \mathbf{X}_2(s) \).
Will show:
\[ f'(s) = 0. \]
for all \( s \); for then \( f(s) = 3 \) and so \( \mathbf{X}_1(s) = \mathbf{X}_2(s) \), as desired.
Proof of \( f'(s) = 0 \) for all \( s \). (Eric Naslund):
Let \( \kappa(s), \tau(s) \) be the common value of curvature and torsion for the two curves. Let
\[
A(s) := \begin{bmatrix}
0 & \kappa(s) & 0 \\
-\kappa(s) & 0 & \tau(s) \\
0 & -\tau(s) & 0
\end{bmatrix}
\]
Let \( tr \) denote transpose. We will use the fact that \( A(s) \) is anti-symmetric: \( A(s)^{tr} = -A(s) \).
And \( f(s) = \mathbf{X}_1(s) \cdot \mathbf{X}_2(s) \).
\[
f'(s) = \mathbf{X}_1'(s) \cdot \mathbf{X}_2(s) + \mathbf{X}_1(s) \cdot \mathbf{X}_2'(s)
\]
\[
= \mathbf{X}_2(s)^{tr} A(s) \mathbf{X}_1(s) + \mathbf{X}_1(s)^{tr} A(s) \mathbf{X}_2(s)
\]
Use Matrix Facts: \( (AB)^{tr} = B^{tr} A^{tr} \) and for a scalar \( u \), \( u^{tr} = u \).
\[
= \mathbf{X}_2(s)^{tr} A(s) \mathbf{X}_1(s) + (\mathbf{X}_1(s)^{tr} A(s) \mathbf{X}_2(s))^{tr}
\]
\[ = X_2(s)^{tr} A(s) X_1(s) + X_2(s)^{tr} A(s)^{tr} (X_1(s)^{tr})^{tr} \]
\[ = X_2(s)^{tr} A(s) X_1(s) - X_2(s)^{tr} A(s) X_1(s) = 0 \]

This proves the Key Proposition. □

In the case \( \tau_1(s) = \tau_2(s) \) for all \( s \), we are done with the "if" part of the Fundamental Theorem of Space Curves.

If \( \tau_1(s) = -\tau_2(s) \) for all \( s \), then apply a reflection to \( C_2 \) across the plane spanned by \( T(0) \) and \( N(0) \) (i.e., the osculating plane) passing through \( \vec{r}(0) \). This maps \( T(s), N(s) \) of \( C_2 \) to \( T(s), N(s) \) of the new version of \( C_2 \) but maps \( B(s) \) to \(-B(s)\) of the new version; it preserves the curvature everywhere and multiplies the torsion everywhere by \(-1\) (this essentially follows from the proof of the "Only If" part of this theorem (the fundamental theorem of space curves) in Case B). Then apply the Key Proposition to the new version of \( C_2 \).

Remark: It seems that the Key Proposition can also be proven by appealing to the uniqueness theorem for the initial value problem for systems of linear ordinary differential equations. The existence theorem can be used to show that if \( \kappa(s) > 0 \) and \( \tau(s) \) are sufficiently smooth functions then there exists a curve whose curvature and torsion are given by \( \kappa(s) \) and \( \tau(s) \).

Mention non-simple curves.

Kepler’s Laws: Here is a first step:

Proposition: Given a system consisting of one sun and one planet, the orbit of the planet is a planar curve with the sun in the plane.

Proof: Let \( \vec{r}(t) \) be the position vector of the planet at time \( t \) with the sun at the origin.

\[ (\vec{r}(t) \times \vec{v}(t))' = \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t) = \vec{r}(t) \times \vec{a}(t) \]
By Newton’s second law, \( F = ma \), and Newton’s law of gravitation:

\[
m\ddot{u}(t) = \bar{F} = -\frac{GMm}{r^2} \bar{u}(t)
\]

where \( \bar{u}(t) = \bar{r}(t)/||\bar{r}(t)|| \).

So, \( \bar{r}(t) \) and \( \bar{a}(t) \) are parallel. So,

\[
(\bar{r}(t) \times \bar{v}(t))' = \bar{r}(t) \times \bar{a}(t) = \bar{0}
\]

So, \( \bar{r}(t) \times \bar{v}(t) \) is a constant vector \( \bar{c} \) and \( \bar{c} \) is normal to \( \bar{r}(t) \).

So, the points \( \bar{r}(t) \) lie in the plane orthogonal to \( \bar{c} \) and passing through \( \bar{0} \), i.e. the sun. □

This proof relied on the fact that \( \bar{a}(t)///\bar{r}(t) \), but not on the inverse square form of the law of gravitation.

We will now use the inverse square law to prove:

Kepler’s First Law: Given a system consisting of one sun and one planet, the orbit of the planet is an ellipse, with the sun as one focus of the ellipse.

Remarks: Note that we are trying to solve the ODE:

\[
\dddot{r} = -\frac{GM}{r^3} \bar{r}
\]

This is a 2nd order nonlinear ODE. Most such equations (even 1st order nonlinear ODE’s) do not have a known explicit solution. But the solar system lucked out.

We are solving the 2-body problem. But still today there is no known explicit solution for the 3-body problem (one sun and two planets).
Proof of Kepler: Let $\vec{c} := \vec{r}(t) \times \vec{v}(t)$ be the (constant) vector normal to the plane of the orbit. Write

$$r = r(t) = ||\vec{r}(t)||$$

Lemma 1:

$$\vec{c} = r^2 \vec{u} \times \frac{d\vec{u}}{dt}$$

This is not completely obvious since $r(t)$ need not be constant.

Proof:

$$\vec{c} = r\vec{u} \times \frac{d(r\vec{u})}{dt} = r^2 (\vec{u} \times \frac{d\vec{u}}{dt}) + r\vec{u} \times \frac{dr}{dt} \vec{u} = r^2 (\vec{u} \times \frac{d\vec{u}}{dt})$$

Lemma 2:

$$\vec{a} \times \vec{c} = GM \frac{d\vec{u}}{dt}$$

Proof:

$$\vec{a} \times \vec{c} = -\frac{GM\vec{u}}{r^2} \times (r^2 \vec{u} \times \frac{d\vec{u}}{dt}) = -GM(\vec{u} \times (\vec{u} \times \frac{d\vec{u}}{dt}))$$

By the triple vector product formula (section 10.3, exercise 23):

$$\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z})\vec{y} - (\vec{x} \cdot \vec{y})\vec{z}$$

the above equals:

$$GM(-\vec{u} \cdot \frac{d\vec{u}}{dt})\vec{u} + (\vec{u} \cdot \vec{u})\vec{u}' = GM\vec{u}'$$

($\vec{u} \cdot \frac{d\vec{u}}{dt} = 0$ since $|\vec{u}|$ is constant). $\Box$

It follows from Lemma 2 that

$$\frac{d(\vec{v} \times \vec{c})}{dt} = \vec{a} \times \vec{c} = \frac{d(GM\vec{u})}{dt}$$

and so there is a constant vector $\vec{d}$ s.t

$$\vec{v} \times \vec{c} = GM\vec{u} + \vec{d}$$
Lecture 11:

Recall:

Recall Notation: \( \vec{r}(t), \vec{v}(t), \vec{a}(t) \) denote position, velocity, acceleration vectors of a planet, with the sun at the origin. Write

\[
r = r(t) = |\vec{r}(t)|, \quad \text{so} \quad \vec{u} = \vec{r}/r
\]

Recall: Proposition: Given a system consisting of one sun and one planet, the orbit of the planet is a planar curve with the sun in the plane.

Recall: using Newton's \( F = ma \) and law of gravitation, we showed that \( \vec{c} := \vec{r}(t) \times \vec{v}(t) \) is constant and is the vector normal to the plane of the orbit.

Will prove:

Kepler's First Law: Given a system consisting of one sun and one planet, the orbit of the planet is an ellipse, with the sun as one focus of the ellipse.

Proof of Kepler:

Lemma 1:

\[
\vec{c} = r^2 \vec{u} \times \vec{u}'
\]

This is not completely obvious since \( r(t) \) need not be constant.

Proof: last time.

Lemma 2:

\[
\vec{a} \times \vec{c} = GM \vec{u}'
\]

Proof:

\[
\vec{a} \times \vec{c} = -\frac{GM \vec{u}}{r^2} \times (r^2 \vec{u} \times \vec{u}') = -GM(\vec{u} \times (\vec{u} \times \vec{u}')).
\]
Since \(|\vec{u}| = 1\) is constant, \(\vec{u} \perp \vec{u}'\). Thus, \(\vec{u}, \vec{u}', \vec{u} \times \vec{u}'\) form an orthogonal system. And so
\[
\vec{u} \times (\vec{u} \times \vec{u}') = -\vec{u}'
\]
because
\[
|\vec{u} \times (\vec{u} \times \vec{u}')| = |\vec{u}||\vec{u}'| = |\vec{u}'|
\]
and the right hand rule gives \(-\) sign. □

By the triple vector product formula (section 10.3, exercise 23),
\[
\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}
\]
the above equals:
\[
GM(-\langle \bar{u} \cdot \bar{u}' \rangle \bar{u} + \langle \bar{u} \cdot \bar{u} \rangle \bar{u}') = GM\bar{u}'
\]
\((\bar{u} \cdot \bar{u}' = 0\) since \(|\vec{u}| = 1\) is constant).)

It follows from Lemma 2 that
\[
(\bar{v} \times \bar{c})' = \bar{a} \times \bar{c} = GM\bar{u}'
\]
and so there is a constant vector \(\bar{d}\) s.t
\[
\bar{v} \times \bar{c} = GM\bar{u} + \bar{d}
\]
Moreover, since \(\bar{u}\) and \(\bar{v} \times \bar{c}\) lie in the plane of the orbit, \(\bar{d}\) must lie in that plane.
If \( \vec{d} = \vec{0} \), then \( \vec{v} \perp \vec{u} \) and so \( \vec{v} \perp \vec{r} \) and so \( |\vec{r}| \) is constant. Thus, \( \vec{r}(t) \) lies in sphere centered at the origin and a plane through the origin and so \( \vec{r}(t) \) traces out a circle, a special ellipse, with the sun at the center.

If \( \vec{d} \neq \vec{0} \), then by an innocent change of coordinates we may assume that the plane of the orbit is the \( xy \)-plane and \( \vec{d} = d\hat{d} \) where \( d = |\vec{d}| > 0 \).

Let \( \theta = \theta(t) \) be the polar angle between \( \vec{d} \) and \( \vec{r}(t) \) (the ordinary polar angle)

\[
\begin{array}{c}
\text{\( \vec{r}(t) \)}
\end{array}
\]

Then \( (r, \theta) = (r(t), \theta(t)) \) are the polar coordinates of \( \vec{r}(t) \).

Letting \( c = |\vec{c}| \), we see that

\[
c^2 = \vec{c} \cdot \vec{c} = \vec{c} \cdot (\vec{r} \times \vec{v}).
\]

By the determinant formula for cross product, we see that this

\[
= \vec{r} \cdot (\vec{v} \times \vec{c}) = r\vec{u} \cdot (\vec{v} \times \vec{c}) = r\vec{u} \cdot (GM\vec{u} + \vec{d}) = GMr + rd\cos(\theta)
\]

So in polar coordinates we get that the orbit of the planet is the curve:

\[
c^2 = GMr + rd\cos(\theta)
\]

Equivalently, in rectangular coordinates

\[
c^2 = GM \sqrt{x^2 + y^2 + dx}.
\]
Equivalently,

\[ c^4 - 2c^2 dx + d^2 x^2 = (GM)^2 (x^2 + y^2), \]

a quadratic. Now with some manipulations, letting \( e := d/(GM) \), we see that if \( |e| < 1 \), we get an ellipse, if \( e = \pm 1 \), we get a parabola and if \( |e| > 1 \), we get a hyperbola. But hyperbolas and paraolas are unbounded. So, it must be an ellipse!

Starting Chapter 15.

Defn: A vector field is a function: \( \vec{F} : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) where \( U \) is a subset of \( \mathbb{R}^n \). The set \( U \), often called a domain, is usually open and often all of \( \mathbb{R}^n \).

\( \vec{x} \in U \) is viewed as a point in \( \mathbb{R}^n \) and \( \vec{F}(\vec{x}) \) is viewed as a vector in \( \mathbb{R}^n \) whose tail is at \( \vec{x} \).

We often write \( \vec{F} = (F_1, \ldots, F_n) \) in terms of its real-valued component functions.

Example: \( \vec{F} \) is a velocity field: \( \vec{F}(\vec{x}) \) is the velocity (direction and magnitude) of a fluid at point \( \vec{x} \). Tells how fluid is flowing at each point in domain.

We usually assume that \( \vec{F} \) is \( C^2 \), but sometimes \( C^1 \) will do.

Examples in \( \mathbb{R}^2 \):
\( F(x, y) = (x, y) \):

\( F(x, y) = (-x, -y) \):
$F(x, y) = (x, -y)$:

$F(x, y) = (y, -x)$:

$F(x, y) = (y, x)$:

Examples in $\mathbb{R}^3$: $F(\bar{x}) = \frac{-GM\bar{x}}{||\bar{x}||^3} = -\frac{GM}{||\bar{x}||^2} \frac{\bar{x}}{||\bar{x}||}.$

Defn: A flow line for a vector field is a parameterized curve $\bar{r}(t)$ satisfying the ODE:

$$\bar{r}'(t) = \bar{F}(\bar{r}(t))$$

In other words a parameterized curve s.t. at the point $\bar{r}(t)$, the tangent vector is parallel to $\bar{F}(\bar{r}(t))$ and the speed is $||\bar{F}(\bar{r}(t))||$.

Visualize the flow lines for the vector fields in the examples above.