HW1 has been posted.

Facts about space-filling curves: a space filling, or square filling, (parameterized) curve can be continuous (e.g., Hilbert curve) or 1-1 but not both; it cannot be differentiable.

Note: We will sometimes view \( \mathbf{r}(t) \) as a point in \( \mathbb{R}^n \) and sometimes as a vector (the position vector) in \( \mathbb{R}^n \).

Review dot product and cross product of vectors.

1. Dot Product \( \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i \)

2. Cross Product (defined in only in \( \mathbb{R}^3 \))

\[
\mathbf{x} \times \mathbf{y} = \det \begin{bmatrix}
\hat{i} & \hat{j} & \hat{k} \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{bmatrix}
\]

Some important facts (Review them now!)

1. \( \mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| ||\mathbf{y}|| \cos(\theta) \)

2. \( \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \)

3. \( \mathbf{x} \perp \mathbf{y} \) iff \( \mathbf{x} \cdot \mathbf{y} = 0 \).

4. \( ||\mathbf{x}|| := \sqrt{\mathbf{x} \cdot \mathbf{x}} \)

5. \( \mathbf{x} \times \mathbf{y} \perp \mathbf{x}, \mathbf{y} \) (direction of \( \mathbf{x} \times \mathbf{y} \) determined by right-hand rule)

6. \( ||\mathbf{x} \times \mathbf{y}|| = ||\mathbf{x}|| ||\mathbf{y}|| \sin(\theta) \)

7. \( \mathbf{x} \) and \( \mathbf{y} \) are parallel iff \( \mathbf{x} \times \mathbf{y} = \mathbf{0} \).

8. \( \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x} \)

9. \( \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j} \)
Theorem 1 (p. 633): (Product Rule) For differentiable parametrized curves $\mathbf{x}(t), \mathbf{y}(t)$

1. $(\mathbf{x} \cdot \mathbf{y})' = \mathbf{x}' \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}'$

2. in $\mathbb{R}^3$:

   $(\mathbf{x} \times \mathbf{y})' = (\mathbf{x}') \times \mathbf{y} + \mathbf{x} \times \mathbf{y}'$

See also Theorem 1 for other differentiation rules.

Proof of 1: extend product rule from one-variable calculus:

$$(\mathbf{x} \cdot \mathbf{y})' = \left( \sum_{i=1}^{n} x_i y_i \right)' = \sum_{i=1}^{n} (x_i y_i)'$$

$$= \sum_{i=1}^{n} x'_i y_i + x_i y'_i = (\mathbf{x}') \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}'. $$

Outline of Proof of 2:

Since $\mathbf{x} \times \mathbf{y}$ is defined as a determinant, it is the signed sum of terms of the form $x_u y_v \overline{m}$, where $\overline{m} = \overline{i}, \overline{j}$ or $\overline{k}$ and $u, v \in \{1, 2, 3\}$. The derivative of such a term is:

$$(x'_u y_v \overline{m} + x_u (y'_v) \overline{m}$$

a sum of two sub-terms. Collecting all the first sub-terms, we get $(\mathbf{x}') \times \mathbf{y}$. Collecting all the second sub-terms, we get $\mathbf{x} \times (\mathbf{y}')$.

Complete proof in HW1.

Proposition: For a differentiable curve $\mathbf{r}(t), ||\mathbf{r}(t)||$ is constant iff $\mathbf{r}(t) \perp \mathbf{v}(t)$.

Note: here we are viewing $\mathbf{r}(t)$, as a vector.

Proof:
\[ ||\vec{r}(t)|| \text{ is constant iff } \vec{r}(t) \cdot \vec{r}(t) \text{ is constant. And} \]
\[ (\vec{r}(t) \cdot \vec{r}(t))' = \vec{r}(t)' \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}(t)' = 2\vec{r}(t)' \cdot \vec{r}(t) \]

by symmetry. Thus, LHS is 0 iff RHS is 0.

· Special case: circle centered at the origin,

Note that \[ ||\vec{r}(t)|| \text{ is constant iff its image curve lies on a sphere centered at the origin.} \]

Compare with:

Corollary (Example 6, p. 634, in text): For a twice differentiable curve, \[ ||\vec{v}(t)|| \text{ is constant iff } \vec{v}(t) \perp \vec{a}(t). \]
A parametric curve $\vec{r}(t)$ is smooth if:
- $C^1$-and-
- $\vec{r}'(t)$ is never $0$.

A curve is smooth if it has a smooth parametrization.

Intuitively, smoothness of a curve means:
- no discontinuities, no sharp corners, no cusps.

Examples:

Smooth:

\[ \begin{array}{ccc}
\text{smooth} & & \text{smooth} \\
\text{non-smooth} & & \text{non-smooth}
\end{array} \]

How should we define the arc length of a smooth parameterized curve $\vec{r} : [a, b] \rightarrow \mathbb{R}^n$? Say for $n = 3$:

Partition curve into sub-curves: $a = t_0 < t_1 \ldots < t_k = b$. 

\[ \vec{r}(t_0), \vec{r}(t_1), \ldots, \vec{r}(t_k) \]
Length of curve is approximately the sum of lengths of chords:

\[
\sum_{i=1}^{k} \| \bar{r}(t_i) - \bar{r}(t_{i-1}) \| = \sum_{i=1}^{k} \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2} = \\
\sum_{i=1}^{k} \sqrt{\left( \frac{\Delta x_i}{\Delta t_i} \right)^2 + \left( \frac{\Delta y_i}{\Delta t_i} \right)^2 + \left( \frac{\Delta z_i}{\Delta t_i} \right)^2} \Delta t_i,
\]

where \( \bar{r}(t) = (x(t), y(t), z(t)) \) and \( \Delta t_i = t_i - t_{i-1}, \Delta x_i = x(t_i) - x(t_{i-1}), \Delta y_i = y(t_i) - y(t_{i-1}), \Delta z_i = z(t_i) - z(t_{i-1}). \)

Define the arc length to be the limit of these sums as the \( \Delta t_i \to 0. \) But this is a limit of Riemann sums. So, we define the arc length of a smooth parametrized curve \( \bar{r} : [a, b] \to \mathbb{R}^n \) as:

\[
L(\bar{r}) = \int_a^b \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \, dt = \int_a^b \| \bar{r}'(t) \| \, dt
\]

This is analogous to the idea that Distance = Speed \times Time.

Why do we need \( C^1? \)

Because we need continuity in order to integrate.

Why do we want smooth?

One reason: so that the parametrization can’t suddenly stop and turn around.

The arc length parameter of a \( C^1 \) parameterized curve is defined

\[
s(t) = \int_a^t \| \bar{r}'(u) \| \, du
\]

Proposition: For a smooth parameterized curve, \( s(t) \) is a 1-1 function and thus has an inverse function \( t(s) \). Moreover, \( s(t) \) and \( t(s) \) are \( C^1 \) functions and \( t'(s) = 1/s'(t) \)
Proof: By FTC, since $\mathbf{r}(t)$ is smooth, $s'(t) = ||\mathbf{r}'(t)|| > 0$. So, $s(t)$ is strictly increasing and thus is a 1-1 function. The Moreover follows from the one-dimensional inverse function theorem. In particular, for the last statement of the proposition, since $s(t)$ and $t(s)$ are inverse functions, we have

$$t(s(t)) = t$$

and so by the chain rule $t'(s(t))(s'(t)) = 1$ and so $t'(s) = 1/s'(t)$. $\square$
MATH 227
Lecture 4: Smooth

Recall for a $C^1$ parameterized curve,

$$s(t) = \int_a^t ||\vec{r}'(u)|| \, du$$

Proposition: For a smooth parameterized curve, $s(t)$ is a 1-1 function and thus has an inverse function $t(s)$. Moreover, $s(t)$ and $t(s)$ are $C^1$ functions and $t'(s) = 1/s'(t)$.

In the last statement of the proposition, we are using $s$ and $t$ as both functions and as variables. A more careful proof of this statement: write $s = f(t) = \int_a^t ||\vec{r}'(u)|| \, du$, and $t = f^{-1}(s)$, Then $f^{-1}(f(t)) = t$ and so $(f^{-1})'(f(t))(f'(t)) = 1$ and so $(f^{-1}(f(t)))' = 1/f'(t)$ and so $t'(s) = 1/s'(t)$.

Note: $s(t)$ is how far you have travelled along the curve after time $t$ (according to the original parameterization $\vec{r}(t)$). And $t(s)$ is the time it takes (in the original parameterization) for you to travel length $s$.

The **arc length parameterization** of a smooth parametrized curve is defined:

$$\vec{x}(s) = \vec{r}(t(s))$$

e.g, $\vec{x}(2)$ is the point along the curve traced out by $\vec{r}$ that is 2 units away from the base point $\vec{r}(a)$ (in the direction given by $\vec{r}$).

By the chain rule,

$$\vec{x}'(s) = \vec{r}'(t(s))t'(s) = \vec{r}'(t)/s'(t) = \vec{r}'(t)/||\vec{r}'(t)||$$

a unit vector. So,

$$||\vec{x}'(s)|| = 1$$
and so the arclength parametrization travels at **unit speed**. This makes sense: if we regard a parameter (in this case $s$) as time then, then time $=$ distance travelled along the curve, and so speed $= 1$ ($ds/ds = 1$).

The arclength parameterization is just a different parameterization of the curve produced by the original paramaterization: they trace out the same physical curve, with the same orientation.

Note: in the text, $\vec{x}(s)$ is written as $\vec{r}(s)$ which is an abuse of notation. We will follow this convention.

$$||\vec{r}'(s)|| = 1$$

Examples (assume $a = 0$):

1. straight line:
   
   $\vec{r}(t) = \vec{u} + t\vec{v}$ \quad ($\vec{v} \neq \vec{0}$)
   
   $\vec{r}'(t) = \vec{v}$
   
   $s(t) = ||\vec{v}||t$
   
   $t(s) = s/||\vec{v}||$
   
   $\vec{r}(s) = \vec{u} + s\vec{v}/||\vec{v}||$ (unit speed)

2. circle $\vec{r} : [0, 2\pi) \rightarrow \mathbb{R}^2$, $\vec{r}(t) = (\rho \cos(t), \rho \sin(t))$,

   $\vec{r}'(t) = (-\rho \sin(t), \rho \cos(t))$ and so $||\vec{r}'(t)|| = \rho$. Thus,

   $$s(t) = \int_0^t ||\vec{r}'(u)||du = \rho t$$

   (Note: $s(2\pi) = 2\pi \rho$).

   The arc length parametrization is:

   $$\vec{r}(s) = (\rho \cos(s/\rho), \rho \sin(s/\rho))$$
\[ ||\vec{r}'(s)|| = ||(-\sin(s/\rho), \cos(s/\rho))|| = 1. \]

3. helix (Example 7, p. 643)

\[ \vec{r}(t) = (a \cos(t), a \sin(t), bt), \quad \vec{r}'(t) = (-a \sin(t), a \cos(t), b) \]

and so \[ ||\vec{r}'(t)|| = \sqrt{a^2 + b^2} \] and so

\[ s(t) = t\sqrt{a^2 + b^2} \]

The arc length parametrization is:

\[ \vec{r}(s) = (a \cos(s/\sqrt{a^2 + b^2}), a \sin(s/\sqrt{a^2 + b^2}), bs/\sqrt{a^2 + b^2}) \]

Note:

\[ ||\vec{r}'(s)|| = ||(-a/\sqrt{a^2 + b^2}) \sin(s/a), (a/\sqrt{a^2 + b^2}) \cos(s/a), b/\sqrt{a^2 + b^2})|| = 1. \]

4. \( \vec{r}(t) = (e^t \cos t, e^t \sin t) \): spiral in the plane

\[ \vec{r}'(t) = (-e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t) \]

\[ ||\vec{r}'(t)|| = e^t \sqrt{(-\sin t + \cos t)^2 + (\sin t + \cos t)^2} = \sqrt{2}e^t. \]

\[ s(t) = \int_0^t \sqrt{2e^u} du = \sqrt{2}(e^t - 1). \]

\[ t(s) = \ln\left(\frac{s}{\sqrt{2}} + 1\right). \]

\[ \vec{r}(s) = \left(\frac{s}{\sqrt{2}} + 1\right) \cos(\ln\left(\frac{s}{\sqrt{2}} + 1\right)), \left(\frac{s}{\sqrt{2}} + 1\right) \sin(\ln\left(\frac{s}{\sqrt{2}} + 1\right)) \right) \]

Can we make sense of \( \vec{r}(s) \) if \( s < 0 \)? Yes, just travel from \( \vec{r}(0) = (1, 0) \) in the reversed orientation for length \( |s| \).
What happens if $s < -\sqrt{2}$???

arc length of curve from $t = -\infty$ to $t = 0$ (i.e. the point $(1,0)$:

$$\int_{-\infty}^{0} \sqrt{2}e^{u}du = \sqrt{2}.$$  

5. Example where you cannot even find $s(t)$ explicitly 

$$\vec{r}(t) = (t, t^2, t^3)$$

$$s(t) = \int_{0}^{t} \sqrt{1 + 4u^2 + 9u^4}du$$

no explicit formula.

For this reason, we cannot always rely on the arclength parametrization, in practice.

HW: For a smooth curve, the arc length parametrization is unique once we specify the base point $\vec{r}(a)$ and the orientation.

Examples of non-smooth curves:

1. $\{(x, y) : y = |x|\}$: A continuous but not differentiable parameterization: $f(t) = (t, |t|)$.

2. $\{(x, y) : y = f(x), x \neq 0\} \cup \{(0,0)\}$ where

$$f(x) = \begin{cases} 
  t^2 \sin(1/t) & t \neq 0 \\
  0 & t = 0 
\end{cases}$$

A differentiable but not $C^1$ parameterization: $\vec{r}(t) = (t, f(t))$

3. $\{(x, y) : y = x^{2/3}\}$. A $C^1$ but not smooth parameterization:

$$\vec{r}(t) = (t^3, t^2)$$

Example: A $C^1$ but not smooth parameterization

$$\vec{r}(t) = (t^3, t^3)$$

But the same curve (a straight line) has a smooth parametrization: $\vec{r}(t) = (t, t)$. 

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MATH 227
Lecture 5:

*Differential Geometry of space curves:*
Let $\vec{r}(t)$ be a smooth parameterized curve (in $\mathbb{R}^2, \mathbb{R}^3$).

Defn: **Unit Tangent vector**

$$\mathbf{T} = \mathbf{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||}$$

For arclength parametrization $\vec{r}(s)$, we have $||\vec{r}'(s)|| = 1$, and so

$$\mathbf{T}(s) = \vec{r}'(s)$$

so, there is no need to normalize.

This is a field of vectors along the curve.

Defn: An **oriented smooth curve** is a smooth curve together with a choice of orientation.

Fact: $\mathbf{T}(s)$ is **intrinsic** in that it depends only on the underlying oriented smooth curve, not on any specific parameterization (strictly speaking, it also depends on a choice of starting point).

**Curvature:**

We will need to assume more than smoothness: *Assume $\mathbf{T}(s)$ is differentiable.*

Defn. of **curvature:**

$$\kappa(s) = ||\frac{d\mathbf{T}}{ds}||$$

Curvature is the rate of “turning of the tangent line to the curve” at a given point on the curve.

More discussion later.
Note: \( \kappa(s) \geq 0 \).

Note: since \( ||\mathbf{T}(s)|| = 1, \) \( d\mathbf{T}/ds \perp \mathbf{T} \).

Defn: If \( \kappa(s) \neq 0 \), then principal normal

\[
\mathbf{N} = \mathbf{N}(s) := \frac{d\mathbf{T}}{\kappa ds}
\]

another field of vectors along the curve. If \( \kappa(s) = 0 \), then we don’t define \( \mathbf{N}(s) \).

Equivalently,

\[
\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s)
\]

Note that \( \mathbf{T}(s) \) and \( \mathbf{N}(s) \) are unit vectors and are \( \perp \). So, they form an orthonormal basis for the plane that they span, called the osculating plane. This is the plane that “best fits” the curve in a nbhd. of a point \( \overline{r}(s) \) on the curve. We can a “field” of plane along the curve.

Note: \( \mathbf{T}(s), \mathbf{N}(s), \kappa(s) \) are intrinsic to an oriented smooth curve.

Examples (\( \overline{r}(s) \) is arclength parameterization):

1. straight line:

\[
\overline{r}(s) = \overline{u} + s\overline{v}/||\overline{v}||
\]

\[
\mathbf{T}(s) = \overline{r}'(s) = \overline{v}/||\overline{v}||
\]

\[
d\mathbf{T}/ds = 0 \perp \mathbf{T}(s)
\]

\[
\kappa(s) = ||d\mathbf{T}/ds|| = 0.
\]

\( \mathbf{N} \) not defined.

2. Circle:

\[
\overline{r}(s) = (\rho \cos(s/\rho), \rho \sin(s/\rho)).
\]
\[ \mathbf{T}(s) = \bar{r}'(s) = (-\sin(s/\rho), \cos(s/\rho)) \]
\[ d\mathbf{T}/ds = (1/\rho)(-\cos(s/\rho), -\sin(s/\rho)) \perp \mathbf{T}(s) \]
\[ \kappa(s) = ||d\mathbf{T}/ds|| = 1/\rho \]

Curvature of a small circle is large and curvature of a large circle is small.

\[ \mathbf{N}(s) = (-\cos(s/\rho), -\sin(s/\rho)) \]

3. Helix: Let \( c = \sqrt{a^2 + b^2} \).

\[ \bar{r}(s) = (a \cos(s/\sqrt{a^2 + b^2}), a \sin(s/\sqrt{a^2 + b^2}), bs/\sqrt{a^2 + b^2}) \]
\[ \mathbf{T}(s) = \bar{r}'(s) = (1/\sqrt{a^2 + b^2})(-a \sin(s/\sqrt{a^2 + b^2}), a \cos(s/\sqrt{a^2 + b^2}), b) \]
\[ d\mathbf{T}/ds = (1/(a^2 + b^2))(-a \cos(s/\sqrt{a^2 + b^2}), -a \sin(s/\sqrt{a^2 + b^2}), 0) \perp \mathbf{T}(s) \]
\[ \kappa(s) = ||d\mathbf{T}/ds|| = a/(a^2 + b^2) \]
\[ \mathbf{N}(s) = (-\cos(s/\sqrt{a^2 + b^2}), -\sin(s/\sqrt{a^2 + b^2}), 0) \]

for fixed \( a \), curvature decreases with increasing \( b \)

- for large \( b \), looks like a straight line.
- for small \( b \), looks like a circle.

4. \( \bar{r}(t) = (e^t \cos t, e^t \sin t) \): spiral.

Curvature should be small for points far from origin and large for points close to origin.
Let \( u(s) = \frac{s}{\sqrt{2}} + 1 \).
\[
\bar{r}(s) = (u(s))(\cos(\ln(u(s))), \sin(\ln(u(s)))).
\]
\[
\mathbf{T}(s) = \left( \frac{1}{\sqrt{2}} \right) (\cos(\ln(u(s))) - \sin(\ln(u(s))), \sin(\ln(u(s))) + \cos(\ln(u(s))))
\]
\[
\frac{dT(s)}{ds} = \frac{1}{\sqrt{2}(s+\sqrt{2})} (-\sin(\ln(u(s))) - \cos(\ln(u(s))), \cos(\ln(u(s))) + \sin(\ln(u(s))))
\]
\[
\kappa(s) = \frac{1}{s+\sqrt{2}}
\]