Lecture 23:

Last time, we defined orientation of smooth surface:

Defn: A smooth surface $S$ is **orientable** if there is a continuous unit normal vector field $\mathbf{N}(\mathbf{p})$, $\mathbf{p} \in S$. Sphere, Torus, Plane; **Not** Klein Bottle

Orientation of smooth surface with boundary: same as for smooth surface – ignore surface boundary.

Example of orientable smooth surface with boundary: cylinder (with only sides); non-example: Mobius band.

IMPORTANT: Induced orientation of boundary curves on smooth surface with boundary: choose orientation on curve s.t. if you ride a bicycle along the boundary curve with the vector from your feet to your head matching the normal on the surface, then the surface is on your left.

Orientability of piecewise smooth surface: where the pieces meet, the boundary curves have **opposite** orientations.
Example of orientable piecewise smooth surface: surface of cube; non-example: crinkly Klein bottle.

Defn: Let \( \mathbf{F} \) be a continuous vector field on an oriented surface \( S \). Let \( \mathbf{N} \) be a continuous unit normal vector field on \( S \). The **vector surface integral** (or **flux**) of \( \mathbf{F} \) across \( S \) is defined:

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \mathbf{N} \, dS
\]

note that the RHS is a scalar surface integral.

Motivation: \( \mathbf{F} \) is a velocity fluid flow. The flux represents the rate of fluid flow across \( S \).

Or a force field.

Example 1: Let \( S \) be the sphere of radius \( \rho \) centered at \( \mathbf{0} \).

\[
\mathbf{F} = \frac{\mathbf{x}}{||x||^3}.
\]

Find flux of \( \mathbf{F} \) out of \( S \). Use: \( \mathbf{N} = \frac{\mathbf{x}}{||x||} \).

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_S \frac{x}{||x||^3} \cdot \frac{x}{||x||} 
\]

\[
= \frac{1}{\rho^2} \int \int_S dS = \frac{4\pi \rho^2}{\rho^2} = 4\pi.
\]

So, the flux integral does not depend on the radius.

Example 2: Let \( S \) be the cylinder, including top and bottom, \( x^2 + y^2 = \rho, \ -h \leq z \leq h \).

Let \( \mathbf{F} = \mathbf{x} \), which is the outward radial vector.

Find the flux of \( \mathbf{F} \) out of \( S \).

On top, \( \mathbf{N} = (0, 0, 1) \). On bottom \( \mathbf{N} = (0, 0, -1) \). On side

\[
\mathbf{N}(\rho \cos(\theta), \rho \sin(\theta), z) = (\cos(\theta), \sin(\theta), 0)
\]
On top:
\[ \int \int_{\text{top}} \mathbf{F} \cdot d\mathbf{S} = \int \int_{\text{top}} \mathbf{F} \cdot \mathbf{N} dS = \int \int_{D=x^2+y^2 \leq \rho^2} (x, y, h) \cdot (0, 0, 1) dS \]
\[ = h(\text{area of } D) = \pi \rho^2 h \]

By symmetry, \[ \int \int_{\text{bottom}} \mathbf{F} \cdot d\mathbf{S} = \int \int_{\text{top}} \mathbf{F} \cdot d\mathbf{S} = \pi \rho^2 h. \]

On side: \[ dS = dz \rho d\theta = \rho dz d\theta \]
\[ \int \int_{\text{side}} \mathbf{F} \cdot d\mathbf{S} = \int \int_{\text{side}} \mathbf{F} \cdot \mathbf{N} dS \]
\[ = \int_{\theta=0}^{2\pi} \int_{z=-h}^{h} (\rho \cos(\theta), \rho \sin(\theta), z) \cdot (\cos(\theta), \sin(\theta), 0) \rho dz d\theta \]
\[ = \rho^2 \int_{\theta=0}^{2\pi} \int_{z=-h}^{h} dz d\theta = 4\pi \rho^2 h \]

So, total flux is \( 6\pi \rho^2 h. \)

Calculating flux integrals over a parameterized surface: \( \mathbf{r} : D \to S \):
\[ \int \int_{S} \mathbf{F} \cdot d\mathbf{S} = \int \int_{S} \mathbf{F} \cdot \mathbf{N} dS = \int \int_{D} \mathbf{F} \cdot \frac{\mathbf{\bar{n}}}{||\mathbf{\bar{n}}||} ||\mathbf{\bar{n}}|| du dv \]
\[ = \int \int_{D} \mathbf{F} \cdot \mathbf{\bar{n}} du dv = \int \int_{D} \mathbf{F} \cdot \left( \frac{\partial \mathbf{\bar{r}}}{\partial u} \times \frac{\partial \mathbf{\bar{r}}}{\partial v} \right) du dv \]

Proposition: Independence of parameterization:

If two parameterizations induce the same unit normal \( \frac{\mathbf{\bar{n}}}{||\mathbf{\bar{n}}||} \), then they give the same flux integral; if they induce opposite unit normals, then their flux integrals are negatives of one another.

Re-do Example 1 with a parameterization:
Lecture 24:

Recall defn of vector surface integral (a.k.a. flux integral) for an oriented parameterized surface, \( \mathbf{r} : D \to S \subset \mathbb{R}^n \)

\[
\mathbf{W} = \langle \mathbf{n} \rangle
\]

\[
\iiint_S \mathbf{F} \cdot d\mathbf{S} := \int \int_S \mathbf{F} \cdot \mathbf{N} \ dS = \int \int_D \mathbf{F}(u, v) \cdot \mathbf{n} \ dudv
\]

\[
= \int \int_D \mathbf{F}(u, v) \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv
\]

Note similarity with vector line integrals:

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt
\]

Proposition: Independence of parameterization:

If two parameterizations for the same underlying surface \( S \) induce the same unit normal \( \mathbf{N} := \frac{\mathbf{n}}{||\mathbf{n}||} \), then for every continuous vector field \( \mathbf{F} \) on \( S \), they give the same flux integral; if they induce opposite unit normals, then their flux integrals are negatives of one another.

Recall Example 1 from last time:

\( S \) : sphere of radius \( \rho \) centered at \( 0 \), oriented outward.

\( \mathbf{F} := \frac{\mathbf{x}}{||\mathbf{x}||^3} \) where \( \mathbf{x} = (x, y, z) \).

\[
\int \int_S \mathbf{F} \cdot \mathbf{N} \ dS = 4\pi.
\]

Re-do example with standard parameterization of the sphere. Recall

\[
\mathbf{n} = -\rho \sin(\phi)\mathbf{r} = -\rho \sin(\phi)\mathbf{x}.
\]

But this gives the inward pointing normal. So, the flux is

\[
= \int \int_D \mathbf{F} \cdot (-\mathbf{n}) dudv = \int_0^{2\pi} \int_0^\pi \left( \frac{\mathbf{x}}{||\mathbf{x}||^3} \right) \cdot (\rho \sin(\phi)\mathbf{x}) d\phi d\theta
\]
\[ = \int_0^{2\pi} \int_0^\pi \sin(\phi) d\phi d\theta = 4\pi. \]

Can also re-do Example 2 from last time with a parameterization.

Flux capacitor.

*Grad, Div, Curl:*

The \( \nabla \) operator: \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \) operates on real-valued functions: it maps functions to vector fields by the *gradient* operator:

\[
\nabla f := \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).
\]

\( \nabla \) can also operate on vector fields \( \vec{F} \) in two different ways: it maps vector fields to functions by the *divergence* operator:

\[
\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}
\]

and maps vector fields to vector fields by the *curl* operator, which we introduced earlier:

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial x} - \frac{\partial F_3}{\partial y} \right)
\]

Note that \( \nabla f \) and \( \nabla \cdot \vec{F} \) are defined in all dimensions. But \( \nabla \times \vec{F} \) is defined only in dimensions 2 and 3. In dimension 2:

\[
\nabla \times \vec{F} = (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) \hat{k}
\]

What do these operators measure?

*Gradient:* \( \nabla f \) is in the direction of maximum increase of \( f \) and \( |\nabla f| \) is the rate of maximum increase.
Imagine $\vec{F}$ is fluid flow.

**Divergence:** $\nabla \cdot \vec{F}$ is the rate of spreading of fluid, as you move along flowlines:

- $\nabla \cdot \vec{F} > 0$: fluid is spreading;
- $\nabla \cdot \vec{F} < 0$: fluid is shrinking;
- $\nabla \cdot \vec{F} = 0$: fluid is neither spreading nor shrinking;

Examples:

- $\vec{F} = (x, y); \; \nabla \cdot \vec{F} = 2$

- $\vec{F} = (-x, -y); \; \nabla \cdot \vec{F} = -2$

- $\vec{F} = (x, -y); \; \nabla \cdot \vec{F} = 0$

- $\vec{F} = (y, -x); \; \nabla \cdot \vec{F} = 0$

$n = 2$: $|\nabla \times \vec{F}|$ measures rate of rotation; sign determines clockwise (-) or counterclockwise (+).

Examples:

- $\vec{F} = (x, y); \; \nabla \times \vec{F} = 0$
\[
\vec{F} = (-x, -y); \quad \nabla \times \vec{F} = \vec{0}
\]
\[
\vec{F} = (x, -y); \quad \nabla \times \vec{F} = \vec{0}
\]
\[
\vec{F} = (y, -x); \quad \nabla \times \vec{F} = -2\vec{k}
\]

\(n = 3\): \(\nabla \times \vec{F}\) points in direction of axis of maximum rotation, using right-hand rule, and \(|\nabla \times \vec{F}|\) measures rate of rotation (at maximum).

Theorems 1 and 2 of Section 16.1 give more precise interpretations of divergence and curl.

Theorem 3 of Section 16.2 states several properties of grad, div and curl.

Product rules: Theorem 3(a,b,c), section 16.2:
\(\)
a: \(\nabla(\phi \psi) = \phi \nabla \psi + \psi(\nabla \phi)\)
b: \(\nabla \cdot \phi \vec{F} = (\nabla \phi) \cdot \vec{F} + \phi(\nabla \cdot \vec{F})\)
c: \(\nabla \times \phi \vec{F} = (\nabla \phi) \times \vec{F} + \phi(\nabla \times \vec{F})\)
\(\)
Examples \(n = 3\), \(\vec{x} = (x, y, z)\):
\(\)
1. \(\nabla|\vec{x}| = \vec{x}/|\vec{x}|, \vec{x} \neq \vec{0}\).
2. \(\nabla \cdot \vec{x} = 3, \nabla \times \vec{x} = \vec{0}\).
3. \(\vec{F} = \vec{x}/||\vec{x}||^3, \vec{x} \neq \vec{0}\). Find \(\nabla \times \vec{F}\) and \(\vec{\nabla} \cdot \vec{F}\).
\(\)
Write: \(\vec{x}/||\vec{x}||^3 = (1/||\vec{x}||^3)\vec{1}\cdot\vec{x}\).
\(\)
Observe that
\[
\nabla(1/||\vec{x}||^3) = -3||\vec{x}||^{-4}\nabla||\vec{x}|| = -3||\vec{x}||^{-5}\vec{x}
\]
So
\[
\nabla \times (\vec{x}/||\vec{x}||^3) = \nabla(1/||\vec{x}||^3) \times \vec{x} + (1/||\vec{x}||^3)\nabla \times \vec{x}
\]
\[
= (-3||\vec{x}||^{-5})\vec{x} \times \vec{x} + 1/||\vec{x}||^3(\vec{0}) = \vec{0}
\]
And

\[ \nabla \cdot \left( \frac{x}{|x|^3} \right) = \nabla \left( \frac{1}{|x|^3} \right) \cdot \bar{x} + \frac{1}{|x|^3} \nabla \cdot \bar{x} = (\frac{-3|x|^{-5}}{\bar{x}}) \cdot \bar{x} + \frac{3}{|x|^3} = 0. \]
Lecture 25:

Recall that Theorem 3 in section 16.2 gives lots of identities in the calculus of grad, div and curl.

**Caution:** Do not confuse the divergence $\nabla \cdot \overline{F}$, with $\overline{F} \cdot \nabla = F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}$, which is an operator on real-valued functions:

$$(\overline{F} \cdot \nabla) f = F_1 \frac{\partial f}{\partial x} + F_2 \frac{\partial f}{\partial y} + F_3 \frac{\partial f}{\partial z}$$

where, for example, $F_1 \frac{\partial f}{\partial x}$ means multiplication.

Functions $\xrightarrow{\text{gradient}}$ Vector Fields $\xrightarrow{\text{curl}}$ Vector Fields $\xrightarrow{\text{divergence}}$ Functions

Defn: A vector field $\overline{F}$ is *irrotational* if $\nabla \times \overline{F} = \overline{0}$.

Re-state the necessary condition for a $C^2$ function to be conservative:

Theorem 3(h): For a $C^2$ function $f$, $\nabla \times \nabla f = \overline{0}$, i.e. curl(grad) = $\overline{0}$, i.e., every conservative $C^1$ vector field is irrotational.

Proof: used equality of mixed second order partials of $f$.

Defn: A vector field $\overline{F}$ is *incompressible* (or solenoidal or divergence-free) if $\nabla \cdot \overline{F} = 0$.

Theorem 3(g): If $\overline{F}$ is a $C^2$ vector field, then $\nabla \cdot (\nabla \times \overline{F}) = 0$ (div(curl $\overline{F}$) = 0), i.e., any vector field which is the curl of a $C^2$ vector field is incompressible.

Proof: in textbook; uses equality of mixed partials of components of $\overline{F}$.

Theorems 3g and 3h say that the composition of two successive operators in the diagram above are zero.

Last time we showed directly that $\overline{F} = \overline{x}/|\overline{x}|^3$ is irrotational and incompressible.
But we already knew it is irrotational: we showed earlier that $\vec{F}$ is conservative by explicitly constructing a potential ($\phi = -\vec{x}/|\vec{x}|$) for $\vec{F}$; and we know from Theorem 3(h) that a conservative vector field is irrotational.

Converse to Theorem 3(h):

Theorem 4, section 16.2 (special to $\mathbb{R}^2$ and $\mathbb{R}^3$): Every $C^1$ irrotational vector field $\vec{F}$ on a simply connected domain is conservative, i.e., is the gradient of a function.

We proved this for the open unit ball and the text proved it for starlike domains. We showed earlier, by construction an example, that Theorem 4 is false without the simply connected assumption.

There is also a converse to Theorem 3(g): Theorem 5 below.

Fact: The complement of every connected, bounded, orientable, closed smooth surface (abbreviated closed surface) in $\mathbb{R}^3$ has two components: its inside, which is bounded, and its outside which is unbounded.

Defn: A connected subset $D$ of $\mathbb{R}^3$ is bounding if for every closed surface $S$ in $D$, the inside of $S$ is also contained in $D$.

We are mainly interested in “solid” domains $D$.

Examples:
1. $D =$ closed or open ball in $\mathbb{R}^3$ are simply connected and bounding.
2. \( D = \) a punctured closed or open ball in \( \mathbb{R}^3 \) is simply connected (as we saw before) but not bounding because for a sphere \( S \) in \( D \) that “surrounds” the puncture, the inside of \( S \) is not contained in \( D \).

3. \( D = \) a solid torus is bounding but not simply connected.

4. \( D = \) a punctured solid torus (puncture in the interior of the solid torus) is neither bounding nor simply connected.

Theorem 5 (special to \( \mathbb{R}^3 \)): Every \( C^1 \) incompressible vector field \( \overline{F} \) on a bounding domain is a curl field, i.e., is the curl of a vector field \( \overline{F} = \nabla \times \overline{G} \); and \( \overline{G} \) is called a vector potential for \( \overline{F} \).

Exercise 18 of section 16.2 gives a very partial proof of Theorem 5.

Read Example 1, section 16.1, for a method to find a vector potential \( \overline{G} \) for an incompressible vector field on a bounding domain.

Theorem 1, section 16.1: Interpretation of divergence as infinitesimal flux density:

\[
\nabla \cdot \overline{F}(P) = \lim_{\epsilon \to 0} \frac{3}{4\pi \epsilon^3} \int \int_{S_\epsilon} \overline{F} \cdot d\overline{S}.
\]

where \( S_\epsilon \) is the sphere of radius \( \epsilon \) centered at \( P \), with outward normal.

This will follow from Divergence Theorem (Gauss’s Theorem).

Interpretation: this is the per volume flux coming out of small balls centered at \( P \).

Divergence is positive if there is a net flux out from \( P \) and negative if there is a net flux in.
Theorem 2, section 16.1: Interpretation of curl as \textit{infinitesimal circulation density}:

\[
(\nabla \times \bar{\mathbf{F}}(P)) \cdot \mathbf{N} = \lim_{\epsilon \to 0} \frac{\int_{C_{\epsilon}} \bar{\mathbf{F}} \cdot d\mathbf{r}}{\pi \epsilon^2}
\]

where \(C_{\epsilon}\) is a circle of radius \(\epsilon\) centered at \(P\) and bounding a disk with unit normal \(\mathbf{N}\) (here, \(\mathbf{N}\) is an arbitrary unit vector).

This will follow from Stokes’ Theorem.

Interpretation: the projection of the curl on \(\mathbf{N}\) is the limiting per area circulation around axis \(\mathbf{N}\).

Projection is positive if circulation is flowing counter-clockwise relative to \(\mathbf{N}\) and negative if flowing clockwise.