MATH 227
Lecture 1:
Motivation: *(version of) Fundamental Theorem of Calculus (FTC):*

Let \( f : [a, b] \to \mathbb{R} \) be a continuously differentiable function. Then

\[
\int_{a}^{b} f'(t) \, dt = f(b) - f(a).
\]

- integral of the derivative recovers the original function

Re-write:

\[
\int_{[a,b]} f'(t) \, dt = \int_{\partial[a,b]} f(x).
\]

where \( \partial[a,b] = \{a, b\} \), the boundary of the interval \([a,b]\); RHS is a funny kind of integral;

Integral of \( f' \) in 1 dimension = Integral of \( f \) in 0 dimensions.

Main idea of vector calculus: Generalize FTC to higher dimensions:

**Green's Theorem:** Given a "blob" \( D \subset \mathbb{R}^2 \) with boundary \( \partial D \) and \( F : D \to \mathbb{R}^2 (F(x, y) = (F_1(x, y), F_2(x, y))) \) (component functions) sufficiently smooth, then

\[
\int\int_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy = \int_{\partial D} F \cdot ds.
\]

Multiple Integral (2-dimensional) of \( \text{curl}(F) = \text{Line integral (1-} \)
dimensional) of \( F \).

\( F \): viewed as a vector field, i.e., a vector at each point of \( D \) and the \( \text{curl}(F) \) measures “rotation” of the vector field.

Imagine \( F \) represents fluid flow. Total “rotation” of fluid within domain \( D = \text{total “circulation” around } \partial D \).

Will give precise definitions later. Don’t need to understand this now!

We will go to 3 dimensions and beyond!

Chapter 11: Vector functions and curves

**interval:** open, closed, half-open, infinite, half-infinite;
- usually closed interval \( I = [a, b] \), with endpoints \( a, b \), or half-open \([a, b)\)

**vector function:** \( \vec{r} : I \to \mathbb{R}^n \)
- \( \vec{r}(t) = (r_1(t), \ldots, r_n(t)) \)

at each \( t \), \( \vec{r}(t) \) can be viewed either as a point in \( \mathbb{R}^n \) or as a vector (emanating from the origin); in the text, the vector is called the **position vector** of the point.

In \( \mathbb{R}^3 \), can write \( \vec{r}(t) \) as

\[
\vec{r}(t) = (r_1(t), r_2(t), r_3(t)), \text{ or } \vec{r}(t) = r_1(t)\hat{i} + r_2(t)\hat{j} + r_3(t)\hat{k}.
\]

- these notations emphasize different viewpoints: point -vs- vector.
- but we use them interchangeably.

- almost always require that \( \vec{r} \) is at least **continuous** (i.e., all \( r_i(t) \) are continuous), but usually require more, e.g., **differentiable** (i.e., all \( r_i(t) \) are differentiable) or **continuously differentiable** (\( C^1 \)) (i.e., all \( r_i(t) \) are continuously differentiable).
- will focus on \( n = 2, 3 \)

**curve:** range (or image) of a vector function
- the curve is a physical object (a set of points in $\mathbb{R}^n$)
- the vector function parametrizes (or traces out) the curve.
- the vector function is sometimes called a parameterized curve (sometimes we call it a curve for short)
- if $t$ is regarded as time then a parametrization is regarded as a trajectory: it tells you how to traverse the curve and orients it by direction of increasing $t$ (a curve has two possible orientations).

Note: a curve can have several different parameterizations.

Examples:
Describe curve from a given parameterization:

1. $\vec{r} : \mathbb{R} \to \mathbb{R}^2$, $\vec{r}(t) = \vec{u} + t\vec{v}$ (entire straight line)

2. $\vec{r} : \mathbb{R} \to \mathbb{R}^2$, $\vec{r}(t) = \vec{u} + t^3\vec{v}$ entire straight line (different parametrization)

3. $\vec{r} : \mathbb{R} \to \mathbb{R}^2$, $\vec{r}(t) = \vec{u} - t\vec{v}$ entire straight line (reversed orientation)

4. $\vec{r} : [0, 2\pi) \to \mathbb{R}^2$, $\vec{r}(t) = (\rho \cos(t), \rho \sin(t))$, (circle of radius $\rho$, oriented counter-clockwise).
5. \( \bar{r} : [0, \pi] \rightarrow \mathbb{R}^2, \bar{r}(t) = (\rho \sin(t), \rho \cos(t)), \) (circle of radius \( \rho \), oriented clockwise).

6. \( \bar{r} : [0, 4\pi) \rightarrow \mathbb{R}^2, \bar{r}(t) = (\rho \cos(t), \rho \sin(t)) \) (double-tracing circle of radius \( \rho \)).

7. \( \bar{r} : [0, \pi) \rightarrow \mathbb{R}^2, \bar{r}(t) = (\rho \cos(2t), \rho \sin(2t)) \) (double-speed circle of radius \( \rho \)).

8. \( \bar{r} : [0, 2\pi) \rightarrow \mathbb{R}^2, \bar{r}(t) = (2 \cos(t), 3 \sin(t)); \ 9x^2 + 4y^2 = 36 \) (ellipse)

9. \( \bar{r} : \mathbb{R} \rightarrow \mathbb{R}^2, \bar{r}(t) = (e^t, e^{-t}) \): (branch of hyperbola \( xy = 1 \) in first quadrant)

10. \( \bar{r} : \mathbb{R} \rightarrow \mathbb{R}^3, \bar{r}(t) = (e^t, e^{-t}, t) \): curve rising over branch of hyperbola.

11. \( a > 0, b \in \mathbb{R}, \bar{r} : \mathbb{R} \rightarrow \mathbb{R}^3, \bar{r}(t) = (a \cos(t), a \sin(t), bt) \): (helix)

\(- \) the helix spirals counter-clockwise around a right circular cylinder of radius \( a \), up (right-handed) if \( b > 0 \), down (left-handed)
MATH 227
Lecture 2:

More examples:

- (11) \( a > 0, \ b \in \mathbb{R}, \ \overline{r} : \mathbb{R} \to \mathbb{R}^3, \overline{r}(t) = (a \cos(t), a \sin(t), bt): \) (helix)

- the helix spirals counter-clockwise around a right circular cylinder of radius \( a \): up (right-handed) if \( b > 0 \), down (left-handed) if \( b < 0 \), degenerate if \( b = 0 \); \( b > 0 \) large means that it rises fast; \( b > 0 \) small means that it rises slowly

- (12)

Find parametrization of a given curve:

Intersection of right circular cylinder centered at \((1, 0, 0)\) with radius 1 and sphere centered at origin with radius 2, in upper half-space.

\((x - 1)^2 + y^2 = 1\) and \(x^2 + y^2 + z^2 = 4\),

\(x(t) = \cos(t) + 1, \ y(t) = \sin(t)\) parameterizes the intersection of the cylinder and \(xy\)-plane, \(0 \leq t \leq 2\pi\). Let

\[ z(t) = \sqrt{4 - x^2 - y^2} = \sqrt{2 - 2 \cos(t)} = 2 \sin(t/2) \]
Space-filling curve (e.g., Hilbert curve and Peano curve): there is a continuous vector function \( \bar{r}(t) = (r_1(t), r_2(t)), 0 \leq t \leq 1 \), whose image is an entire filled-in square.

Can such a vector function \( \bar{r}(t) \) be differentiable? one-to-one?

If \( \bar{r}(t) \) represents position of a particle at time \( t \) and \( \bar{r}(t) \) is differentiable, we define the velocity: \( \bar{v}(t) := \bar{r}'(t) = (r_1'(t)\hat{i} + r_2'(t)\hat{j} + r_3'(t)\hat{k}) \).

For \( n = 3 \): we sometimes write \( \bar{v}(t) = r_1'(t)\hat{i} + r_2'(t)\hat{j} + r_3'(t)\hat{k} \).

**speed:** of a differentiable parametrized curve: \( ||\bar{v}(t)|| = ||\bar{r}'(t)|| \)

(recall \( ||\bar{u}|| = \sqrt{\sum_{i=1}^{n} u_i^2} \) is the length of \( \bar{u} \) viewed as a vector)
Similarly, for **twice-differentiable** parametrized curve, we define the **acceleration**: \( \overline{a}(t) := \overline{v}''(t) = \overline{r}''(t) \)

If we draw \( \overline{v}(t_0) \), based at \( \overline{r}(t_0) \) (instead of the origin), then it is tangent to the curve \( \overline{r}(t) \) at \( \overline{r}(t_0) \) and points in the direction of increasing \( t \):

\[
\overline{v}(t_0) = \lim_{\Delta t \to 0} \frac{\overline{r}(t_0 + \Delta t) - \overline{r}(t_0)}{\Delta t}
\]

**tangent line:** to parametrized curve \( \overline{r}(t) \) at \( \overline{r}_0 = \overline{r}(t_0) \):

Let \( \overline{v}_0 = \overline{v}(t_0) \)

\[-z(t) = \overline{r}_0 + t\overline{v}_0 \quad (z(0) = \overline{r}_0)\]

- or with a different parametrization:

\[z(t) = \overline{r}_0 + (t - t_0)\overline{v}_0 \quad (z(t_0) = \overline{r}_0)\]

Note: Don’t confuse parameter \( t \) for parametrized curve with parameter \( t \) for tangent line.

\[||u|| = ||\overline{u}|| = \text{length of vector } \overline{u}\]

**Examples:**

1. circle: \( \overline{x}(t) = (\rho \cos(t), \rho \sin(t)) \), \( \rho > 0 \).

\[\overline{v}(t) = (-\rho \sin(t), \rho \cos(t)), ||\overline{v}(t)|| = \rho \text{, constant speed}\]

\[\overline{a}(t) = (-\rho \cos(t), -\rho \sin(t)),\]

Velocity: tangent to circle
Acceleration: points to center of circle
e.g., at \( t = 0 \), \( \vec{r} = (\rho, 0) \), \( \vec{v} = (0, \rho) \), \( \vec{a} = (-\rho, 0) \)

2. helix: \( \vec{r}(t) = (a \cos(t), a \sin(t), bt) \), \( a > 0, b \in \mathbb{R} \)

\( \vec{v}(t) = (-a \sin(t), a \cos(t), b) \), \( ||\vec{v}(t)|| = \sqrt{a^2 + b^2} \), constant speed

\( \vec{a}(t) = (-a \cos(t), -a \sin(t), 0) \)
(horizonal vector that points to axis of cylinder)

![Figure 4: Helix](image)

3. Generic Example: \( \vec{r}(t) = (t^2, e^{-t}, \sin(2t)) \),

\( \vec{v}(t) = (2t, -e^{-t}, 2 \cos(2t)) \), \( ||\vec{v}(t)|| = \sqrt{4t^2 + e^{-2t} + 4 \cos^2(2t)} \)

\( \vec{a}(t) = (2, e^{-t}, -4 \sin(2t)) \)

Tangent line to \( \vec{r}(t) \) at \( \vec{r}(0) \):

\( \vec{z}(t) = (0, 1, 0) + t(0, -1, 2) \)
4. path of a projectile (in three dimensions, \( \vec{k} \) is vertical unit vector):

\[
\vec{r}(0) = \vec{r}_0, \quad \vec{r}'(0) = \vec{v}(0) = \vec{v}_0, \quad \vec{r}'' = \vec{a} = -g\vec{k}.
\]

Integrate:

\[
\vec{v}(t) = \int \vec{a}(s)ds = -gt\vec{k} + \vec{v}_0
\]

Integrate again:

\[
\vec{r}(t) = \int \vec{v}(s)ds = -(gt^2/2)\vec{k} + \vec{v}_0 t + \vec{r}_0.
\]

Review dot product and cross product of vectors.

1. Dot Product \( \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i \)

2. Cross Product (defined in only in \( \mathbb{R}^3 \))

\[
\vec{x} \times \vec{y} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}
\]

Some important facts:

1. \( \vec{x} \cdot \vec{y} = ||\vec{x}|| ||\vec{y}|| \cos(\theta) \)

2. \( \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} \)

3. \( \vec{x} \perp \vec{y} \) iff \( \vec{x} \cdot \vec{y} = 0 \).

4. \( ||\vec{x}|| := \sqrt{\vec{x} \cdot \vec{x}} \)

5. \( \vec{x} \times \vec{y} \perp \vec{x}, \vec{y} \) (direction of \( \vec{x} \times \vec{y} \) determined by right-hand rule)

6. \( ||\vec{x} \times \vec{y}|| = ||\vec{x}|| ||\vec{y}|| \sin(\theta) \)

7. \( \vec{x} \) and \( \vec{y} \) are parallel iff \( \vec{x} \times \vec{y} = \vec{0} \).