Math 227, Homework #9, due Wednesday, April 4

1. Section 16.4: 4, 13, 17, 23, 24

2. Section 16.5: 2, 4, 7, 11, 13

3. Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where

\[
\mathbf{F} = (e^x \sin(x) - y^2 - z^2, xy + z, xz + y)
\]

and \( C \) is the piecewise smooth curve consisting of two pieces, the first parameterized by

\[
\mathbf{\pi}(t) = (\sin(t), 0, \cos(t)), \quad 0 \leq t \leq \pi/4
\]

followed by the second, parameterized by

\[
\mathbf{\eta}(t) = \left( \frac{\cos(t)}{\sqrt{2}}, \frac{\sin(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad 0 \leq t \leq \pi/2.
\]

Solution:

Here, \( C \) is the curve on the unit sphere that traverses, southward, along the longitude from \((0, 0, 1)\) to \((\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\) followed by the curve on the unit sphere that traverses, eastward, along the latitude from \((\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\) to \((0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\). This can be completed to a closed curve \( C + C' \) where \( C' \) traverses, northward, along the longitude from \((0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) to \((0, 0, 1)\). Thus,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot dS
\]

where \( S \) is the smaller part of the sphere bounded by \( C + C' \) (here, \( S \) is oriented with outward pointing normal and \( C + C' \) is oriented consistently). Now,

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
e^{x^2 \sin(x)} - y^2 - z^2 & xy + z & xz + y
\end{vmatrix} = 3(0, -z, y)
\]

Thus, \( \nabla \times \mathbf{F} \) is orthogonal to \( S \) and so \( \int_S \nabla \times \mathbf{F} \cdot dS = 0 \). Thus,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = - \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_{-C'} \mathbf{F} \cdot d\mathbf{r}
\]

\[
\mathbf{\pi}(t) = (0, \sin(t), \cos(t)), \quad 0 \leq t \leq \pi/4
\]

Thus,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/4} \mathbf{F}(\mathbf{\pi}(t)) \cdot \mathbf{\pi}'(t) dt = \int_0^{\pi/4} \mathbf{F}(\mathbf{\pi}(t)) \cdot (0, \cos(t), -\sin(t)) dt
\]
\[
\begin{align*}
\int_0^{\pi/4} F_2(\pi(t)) \cos(t) - F_3(\pi(t)) \sin(t) \, dt &= \int_0^{\pi/4} \cos^2(t) - \sin^2(t) \, dt \\
&= \int_0^{\pi/4} \cos(2t) \, dt = 1/2.
\end{align*}
\]

4. Let \( \mathbf{F} \) be a \( C^1 \) vector field on a domain \( D \subseteq \mathbb{R}^3 \).

(a) Show that if \( \mathbf{F} \) has both a scalar and vector potential and \( \textit{the vector potential is} \ C^2 \), then \( \mathbf{F} = \nabla \phi \) for some harmonic function \( \phi \).

(b) Show that if \( D \) is bounding and \( \mathbf{F} = \nabla \phi \) for some harmonic function \( \phi \), then \( \mathbf{F} \) has both a scalar and vector potential.

Solution:

a. Note that I added the assumption that the vector potential is \( C^2 \), after the HW was turned in – Sorry!.

\( \mathbf{F} = \nabla \phi \) for some function \( \phi \) (which is automatically \( C^2 \) since \( \mathbf{F} \) is \( C^1 \)) and \( \mathbf{F} = \nabla \times \mathbf{G} \) for some vector field \( \mathbf{G} \) (which would not automatically be \( C^2 \) but it is \( C^2 \) because I added the assumption to the problem). Since \( \mathbf{G} \) is \( C^2 \),

\[
\nabla \cdot \mathbf{F} = \nabla \cdot \nabla \times \mathbf{G} = 0
\]

and so

\[
\nabla^2 \phi = \nabla \cdot \nabla \phi = \nabla \cdot \mathbf{F} = 0.
\]

So \( \phi \) is harmonic. \( \square \)

b. If \( \mathbf{F} = \nabla \phi \), then clearly \( \mathbf{F} \) has a scalar potential.

Since \( \phi \) is harmonic, \( \nabla \cdot \mathbf{F} = \nabla^2 \phi = 0 \). Since the domain is bounding, \( \phi \) has a vector potential.

5. Let \( \mathbf{F} \) be a \( C^1 \) vector field on \( \mathbb{R}^3 \) and \( \mathcal{S} \) be the graph of a \( C^2 \) function \( g(x,y) \) on a domain \( D \in \mathbb{R}^2 \). So, the surface \( \mathcal{S} \) projects onto \( D \) and \( C = \partial \mathcal{S} \) (the surface boundary of \( \mathcal{S} \)) projects onto the boundary curve \( C' = \partial D \).

Let \( \mathbf{G} = (G_1, G_2) \) be the vector field in \( \mathbb{R}^2 \) defined by

\[
G_1(x, y) = F_1(x, y, g(x, y)) + F_3(x, y, g(x, y)) \frac{\partial g}{\partial x}
\]

\[
G_2(x, y) = F_2(x, y, g(x, y)) + F_3(x, y, g(x, y)) \frac{\partial g}{\partial y}
\]

Show that

\[
(\nabla \times \mathbf{G}) \cdot \mathbf{k} = (\nabla \times \mathbf{F}) \cdot \mathbf{n}
\]

with normal \( \mathbf{n} \) induced by the parameterization \((x, y, g(x, y))\) and

\[
\mathbf{G}(x(t), y(t)) \cdot (x'(t), y'(t))
\]

2
\[(x(t), y(t), g(x(t), y(t))) \cdot (x'(t), y'(t), g_xx'(t) + g_yy'(t))\]

where \((x(t), y(t))\) parameterizes \(C'\) and \((x(t), y(t), g(x(t), y(t)))\) parameterizes \(C\).

(recall that this was part of my attempt to motivate the proof of Stokes Theorem by reducing to Green’s Theorem).

Solution:

Note that I added the assumption that \(g\) is \(C^2\) (before the HW was turned in).

This problem is a calculation (it is essentially the calculation in the proof of Stokes Theorem on pp. 940-941.)

Proof that \((\nabla \times \overline{G}) \cdot \mathbf{k} = (\nabla \times \overline{F}) \cdot \mathbf{n}\):

\[
(\nabla \times \overline{G}) \cdot \mathbf{k} = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial z} \frac{\partial g}{\partial x} + \frac{\partial F_3}{\partial z} \frac{\partial g}{\partial x} + \frac{\partial F_1}{\partial y} + \frac{\partial F_3}{\partial z} \frac{\partial g}{\partial y} + F_3 \frac{\partial g}{\partial x} \frac{\partial^2 g}{\partial x \partial y}
\]

after cancellation, using that \(g\) is \(C^2\), this equals

\[
\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \frac{\partial g}{\partial x} - \frac{\partial F_2}{\partial y} \frac{\partial g}{\partial y}\right) \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)
\]

\[
= (\nabla \times \overline{F}) \cdot \mathbf{n}
\]

Proof that

\[
\overline{G}(x(t), y(t)) \cdot (x'(t), y'(t)) = \overline{F}(x(t), y(t), g(x(t), y(t))) \cdot (x'(t), y'(t), g_xx'(t) + g_yy'(t))
\]

The LHS is

\[
\left(F_1(x, y, g(x, y)) + F_3(x, y, g(x, y)) \frac{\partial g}{\partial x}\right) x'(t) + \left(F_2(x, y, g(x, y)) + F_3(x, y, g(x, y)) \frac{\partial g}{\partial y}\right) y'(t)
\]

\[
= F_1(x(t), y(t), g(x(t), y(t)))x'(t) + F_2(x(t), y(t), g(x(t), y(t)))y'(t)
\]

\[
+ F_3(x(t), y(t), g(x(t), y(t))) \frac{\partial g}{\partial x} x'(t) + F_3(x(t), y(t), g(x(t), y(t))) \frac{\partial g}{\partial y} y'(t)
\]

which is the RHS.