Math 227, Homework #7, due Friday, March 9, SOLUTIONS

(1) Let
\[ \pi(u, v) = ((2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v)) \]
defined the domain \( D = \{(u, v) : 0 \leq u \leq 2\pi, \ 0 \leq v \leq 2\pi \}. \)

(a) Show that \( \pi \) is smooth on the interior of \( D \), i.e., that \( \pi \) is \( C^1 \) and the normal \( \mathbf{n} := \frac{\partial \pi}{\partial u} \times \frac{\partial \pi}{\partial v} \neq 0. \)

(b) Show that \( \pi \) is one-to-one on the interior of \( D \) (i.e., on the set \( \{(u, v) : 0 < u < 2\pi, \ 0 < v < 2\pi \} \)).

(c) Find all values of \( u, v, u', v' \) such that \( \pi(u, v) = \pi(u', v') \).

(d) Describe the \( u \)-curves and \( v \)-curves

(e) What surface is this? (i.e., what is the image of \( \pi \) on the domain \( D \))

Solution:

(a): \( \pi \) is \( C^1 \) because \( \sin \) and \( \cos \) are \( C^1 \).

\[ \frac{\partial \pi}{\partial u} = (-2 + \cos(v)) \sin(u), (2 + \cos(v)) \cos(u), 0 \]

\[ \frac{\partial \pi}{\partial v} = (-\sin(v) \cos(u), -\sin(v) \sin(u), \cos(v)) \]

So,

\[ \mathbf{n} := \frac{\partial \pi}{\partial u} \times \frac{\partial \pi}{\partial v} = ((2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), (2 + \cos(v)) \sin(v)) \]

So,

\[ \mathbf{n} := |2 + \cos(v)| = 2 + \cos(v) \in [1, 3] \]

and so \( \mathbf{n} \) is never \( 0. \)

(b) Let \((u, v) \in D \) and \((x, y, z) = \pi(u, v) \). Then \( \sin(v) = z \) and \( \cos(v) = \sqrt{x^2 + y^2 - 2} \). Thus, given \((x, y, z) \) we can uniquely determine \((\cos(v), \sin(v)) \), and hence \( v \) up to \( \pm 2\pi \). Then \( \cos(u) = x/(2 + \cos(v)) \) and \( \sin(u) = y/(2 + \cos(v)) \) and thus we can uniquely determine \((\cos(u), \sin(u)) \), and hence \( u \) up to \( \pm 2\pi \).

Thus, given \((x, y, z) \) we can uniquely determine \( u \) and \( v \), each up to \( \pm 2\pi \).

This implies that \( \pi \) is one-to-one on the interior of \( D \).

(c) By the argument in part (1), if \( \pi(u, v) = \pi(u', v') \) and \((u, v) \neq (u', v') \), then the only possibilities are:

- \{u, u'\} = \{0, 2\pi\} and \( v = v' \)
- \{v, v'\} = \{0, 2\pi\} and \( u = u' \)
- \{u, u'\} = \{0, 2\pi\} and \( \{v, v'\} = \{0, 2\pi\} \).

(d) \( u \)-curves: for constant \( v_0 \), the curve is the intersection of \( x^2 + y^2 = (2 + \cos(v_0))^2 \) and \( z = \sin(v_0) \), which is a circle centered at \((0, 0, \sin(v_0)) \) of radius \( 2 + \cos(v_0) \) in the plane \( z = \sin(v_0) \).

\( v \)-curves: for constant \( u_0 \), the curve is the intersection of \((x - 2 \cos(u_0))^2 + (y - 2 \sin(u_0))^2 + z^2 = 1 \) and \( y = \tan(u_0)x \) which is a circle centered at \((2 \cos(u_0), 2 \sin(u_0), 0) \) of radius 1 in the plane \( y = \tan(u_0)x \) (when \( \tan(u_0) = \infty \), this is the plane \( x = 0 \)).

(e) By the result of part (2), the image of \( D \) is obtained by identifying the left and right sides with one another and the top and bottom sides with one another (without a twist). By part (1), there are no other identifications. Thus, we get a torus.
The $u$-curves are circles that go the long way around the donut hole. The $v$-curves are circles that go the short way around.

(2) Let $G(x,y,z)$ be $C^1$. Assume that $\nabla G(x_0,y_0,z_0) \neq (0,0,0)$. Let $S$ be the level surface $G(x,y,z) = G(x_0,y_0,z_0)$. Show that for a sufficiently small ball $B$ centered at $(x_0,y_0,z_0)$, for the surface $B \cap S$, the area element is at least one of $dS = |\nabla G(x,y,z)|/|G_z(x,y,z)|\,dx\,dy$ or $dS = |\nabla G(x,y,z)|/|G_y(x,y,z)|\,dx\,dz$ or $dS = |\nabla G(x,y,z)|/|G_x(x,y,z)|\,dy\,dz$.

Solution: Since $\nabla G(x_0,y_0,z_0) \neq (0,0,0)$, at least one of its components is nonzero. Without loss of generality, $G_z(x_0,y_0,z_0) \neq 0$. By the implicit function theorem, for a sufficiently small ball $B$ centered at $(x_0,y_0,z_0)$, there is a function $g(x,y)$ such that for $(x,y,z) \in B$, we have $(x,y,z) \in S$ iff $z = g(x,y)$. This means that the projection of $B \cap S$ to the $xy$-plane is one-to-one. Then the result follows by the discussion in class (see the discussion in the middle of page 902).

Section 15.6: 4, 11, 12, 17
Section 16.1: 6, 7, 8, 10