Math 227 HW3 due on Friday, Jan. 26 at beginning of class. SOLUTIONS

1. Sec. 11.4: 5, 6, 7, 8
2. Sec. 11.5: 4, 6, 8, 19, 25
3. True or False: for a differentiable vector function \( \mathbf{r}(t) \),
   \[ ||\mathbf{r}'|| = ||\mathbf{r}||' \]
   False: let \( \mathbf{r} \) be the arclength parameterization of the unit circle. Then the RHS is 0 and the LHS is 1.

4. Complete the proof of Theorem 2, page 653-654, by showing that
   \[ \lim_{\Delta s \to 0} \frac{||\Delta \mathbf{T}||}{\Delta \theta} = 1. \]
   Hint: Use law of cosines and L’Hopital’s rule.

Law of cosines:
   \[ C^2 = A^2 + B^2 - 2AB \cos(\theta). \]

Consider the triangle determined by the vectors \( T(s_0) \) and \( T(s_0 + \Delta s) \) emanating from the origin, as drawn in Figure 11.15 of the text (this figure is a copy of the unit disk: the intersection of the plane spanned by \( T(s_0) \) and \( T(s_0 + \Delta s) \) and the unit sphere).

By law of cosines,
   \[ ||\Delta \mathbf{T}||^2 = ||T(s_0)||^2 + ||T(s_0 + \Delta s)||^2 - 2||T(s_0)||||T(s_0 + \Delta s)|| \cos(\Delta \theta) \]
   \[ = 2(1 - \cos(\Delta \theta)) \]
   Thus, by l’Hospital’s rule,
   \[ \lim_{\Delta \theta \to 0} \frac{||\Delta \mathbf{T}||}{\Delta \theta} = \lim_{\Delta \theta \to 0} \sqrt{\frac{2(1 - \cos(\Delta \theta))}{\Delta \theta^2}} \]
   \[ = \lim_{\Delta \theta \to 0} \sqrt{\frac{2 \sin(\Delta \theta)}{2 \Delta \theta}} = \lim_{\Delta \theta \to 0} \sqrt{\frac{2 \cos(\Delta \theta)}{2}} = 1. \]
   Note that \( \Delta \theta \to 0 \) iff \( \Delta s \to 0 \).

One student showed me a simpler solution:
   Let \( h \) be the length of a perpendicular from the head of \( T(s_0 + \Delta s) \) dropped to the base \( T(s_0) \). Then \( h = \sin(\Delta \theta) \) and so
   \[ \sin(\Delta \theta) \leq ||\Delta \mathbf{T}|| \leq \Delta \theta \]
   the latter inequality because the length of the arc of the unit circle subtended by angle \( \Delta \theta \) is \( \Delta \theta \). Now, dividing the inequalities above by \( \Delta \theta \) we get
   \[ \frac{\sin(\Delta \theta)}{\Delta \theta} \leq \frac{||\Delta \mathbf{T}||}{\Delta \theta} \leq \frac{\Delta \theta}{\Delta \theta} = 1. \]
   Now, use the fact that
   \[ \lim_{\Delta \theta \to 0} \frac{\sin(\Delta \theta)}{\Delta \theta} = 1 \]
   and apply the squeeze theorem.