Non-textbook problems are worth more than textbook problems.

1. Section 11.3: 13, 17, 24, 25, 27

2. Find the unit tangent $T$, the unit principal normal $N$ and the curvature $\kappa$ at all points of the parameterized curves in problems 24 and 25 of section 11.3.

Solution: Here, we use the original parameterization because the arclength parameterization is complicated.

For problem 24:

$$\vec{r}' = (e^t, \sqrt{2}, e^{-t}), \quad |\vec{r}'| = \sqrt{2 + e^{2t} + e^{-2t}}$$

and so

$$T = (1/\sqrt{2 + e^{2t} + e^{-2t}})(e^t, \sqrt{2}, e^{-t})$$

$$\vec{r}'' = (e^t, 0, -e^{-t}),$$

$$\vec{r}' \times \vec{r}'' = (-\sqrt{2}e^{-t}, 0, -\sqrt{2}e^t)$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{2(e^{2t} + e^{-2t})}$$

So,

$$\kappa = \frac{\sqrt{2(e^{2t} + e^{-2t})}}{(2 + e^{2t} + e^{-2t})^{3/2}}$$

And

$$B = \frac{(-\sqrt{2}e^{-t}, 0, -\sqrt{2}e^t)}{(2(e^{2t} + e^{-2t}))^{3/2}}$$

And

$$N = B \times T = \frac{(2e^t, \sqrt{2}e^{-2t} - \sqrt{2}e^{2t}, 2e^{-t})}{(2(e^{2t} + e^{-2t}))^{3/2}(2 + e^{2t} + e^{-2t})}$$

For problem 25: this is too computationally complicated for me, but I am sure you can do it, using the same procedure as for problem 24.

3. Let $C$ be the curve $\{(x, y) : y = x^{1/3}, x \in \mathbb{R}\}$.

(a) Find a $C^1$ but non-smooth parameterization of $C$.

(b) Find a smooth parameterization of $C$.

(c) Is $C$ a smooth curve? Why or why not?

Solution:

a. $\vec{r}(t) = (t, t^{1/3})$ is not smooth at $t = 0$ since $t^{1/3}$ is not differentiable at $t = 0$.

b. $\vec{r}(t) = (t^3, t)$ is smooth at $t = 0$ since both component functions are $C^1$ and $r'_2 = 1$ and so $\vec{r}'$ is never $(0, 0)$.

c. The curve is smooth because it has a smooth parameterization.
4. Let $C$ be the curve $\{(x, y) : y = |x|, x \in \mathbb{R}\}$.

(a) Find a $C^1$ parameterization of $C$.

(b) Show that $C$ is not a smooth curve.

Solution:

a. Let $r(t) = (t^3, |t^3|)$. Clearly $t^3$ is $C^1$. And $g(t) = |t^3|$ is clearly $C^1$ at all $t \neq 0$ with $g'(t) = 3t^2$ if $t > 0$ and $g'(t) = -3t^2$ if $t < 0$. So, to show that $g$ is $C^1$ at $t = 0$ it suffices to show that $g'(0) = 0$. Observe

$$\lim_{t \to 0^+} (g(t) - g(0))/t = \lim_{t \to 0^+} t^2 = 0$$

and

$$\lim_{t \to 0^{-}} (g(t) - g(0))/t = \lim_{t \to 0^{-}} -t^2 = 0$$

and so indeed $g'(0) = 0$.

Note: of course $r(t)$ is not a smooth parameterization because $r'(0) = (0, 0)$. The most obvious parameterization of this curve $(t, |t|)$ is not even differentiable.

b. Suppose that $\overline{r}(t) = (r_1(t), r_2(t))$ is a smooth parameterization of $C$. There exists some $t_0$ s.t. $\overline{r}(t_0) = (0, 0)$. Note that $r_2(t) = |r_1(t)|$. We seek a contradiction.

Case 1: $r_1'(t_0) = 0$.

Then

$$r_2'(t_0) = \lim_{t \to t_0^+} r_2(t)/(t - t_0) = \lim_{t \to t_0^+} |r_1(t)/(t - t_0)| = |r_1'(t_0)| = 0$$

and so $\overline{r}'(t_0) = (0, 0)$, a contradiction.

Case 2: $r_1'(t_0) \neq 0$.

WLOG, we may assume that $r_1'(t_0) > 0$. Then,

$$r_2'(t_0) = \lim_{t \to t_0^+} r_2(t)/(t - t_0) = \lim_{t \to t_0^+} |r_1(t)/(t - t_0)| = |r_1'(t_0)| = r_1'(t_0) > 0$$

and

$$r_2'(t_0) = \lim_{t \to t_0^-} r_2(t)/(t - t_0) = \lim_{t \to t_0^-} -|r_1(t)/(t - t_0)| = -|r_1'(t_0)| = -r_1'(t_0) < 0$$

So, $r_2$ has different left and right sided derivatives at $t_0$ and so $r_2$ is not differentiable at $t_0$, a contradiction.