1. Section 17.1: 1, 4, 7
2. Section 17.2: 2, 4, 5, 8, 9
3. Show that a bilinear form \( \eta \) is anti-symmetric (and hence a 2-form) iff there is an anti-symmetric matrix \( A \) such that \( \eta(\bar{x}, \bar{y}) = \bar{x}A\bar{y}^T \) for all vectors \( \bar{x} \) and \( \bar{y} \).

Solution:

*Only if:* Since \( \eta \) is bilinear,

\[
\eta(\bar{x}, \bar{y}) = \eta(\sum_{i=1}^{n} x_i \bar{e}_i, \sum_{j=1}^{n} y_j \bar{e}_j) = \sum_{i=1}^{n} x_i y_j \eta(\bar{e}_i, \bar{e}_j) = \bar{x}A\bar{y}^T
\]

where \( A_{ij} = \eta(\bar{e}_i, \bar{e}_j) \). Since \( \eta \) is anti-symmetric,

\[
A_{ij} = \eta(\bar{e}_i, \bar{e}_j) = -\eta(\bar{e}_j, \bar{e}_i) = -A_{ji}
\]

*If:*

\[
\eta(a\bar{u} + b\bar{v}, \bar{y}) = (a\bar{u} + b\bar{v})A\bar{y}^T = a\bar{u}A\bar{y}^T + b\bar{v}A\bar{y}^T = a\eta(\bar{u}, \bar{y}) + b\eta(\bar{v}, \bar{y})
\]

Similarly,

\[
\eta(\bar{x}, a\bar{u} + b\bar{v}) = \bar{x}A(a\bar{u} + b\bar{v})^T = \bar{x}Aa\bar{u}^T + \bar{x}Ab\bar{v}^T = a\eta(\bar{x}, \bar{u}) + b\eta(\bar{x}, \bar{v})
\]

Thus, \( \eta \) is bilinear.
If \( A \) is anti-symmetric,

\[
\eta(\bar{x}, \bar{y}) = \bar{x}A\bar{y}^T = (\bar{x}A\bar{y}^T)^T = \bar{y}A^T\bar{x}^T = -\bar{y}A\bar{x}^T = -\eta(\bar{y}, \bar{x})
\]

and so \( \eta \) is anti-symmetric.

4. Show that \( \Lambda_k(V) \) (the set of all \( k \) forms on an \( n \)-dimensional vector space \( V \)) is itself a vector space of dimension \( \binom{n}{k} \), with a basis: \( \{ e_{i_1}^* \land e_{i_2}^* \ldots \land e_{i_k}^* : i_1 < i_2 < \ldots < i_k \} \).

Solution: It is straightforward to show that \( \Lambda_k(V) \) is a vector space over \( \mathbb{R} \) (we essentially did that in class). To show that the given set is a basis, we must show that it spans, over \( \mathbb{R} \), all \( k \)-forms and is linearly independent.

*Spanning:*

Let \( \phi \in \Lambda_k(V) \). Since \( \phi \) is \( k \)-linear, for all \( \bar{x}^1, \ldots, \bar{x}^k \in V \),

\[
\phi(\bar{x}^1, \ldots, \bar{x}^k) = \phi(\sum_{j_1=1}^{n} x_{j_1}^1 \bar{e}_{j_1}, \ldots, \sum_{j_k=1}^{n} x_{j_k}^k \bar{e}_{j_k}) = \sum_{j_1, \ldots, j_k=1}^{n} x_{j_1}^1 \ldots x_{j_k}^k \phi(\bar{e}_{j_1}, \ldots, \bar{e}_{j_k})
\]
Since \( \phi \) is anti-symmetric, we may restrict to only those choices of \( j_1, \ldots, j_k \) where the \( j_1, \ldots, j_k \) are distinct. For each such choice, there is a unique reordering of the \( j_1, \ldots, j_k \) as some \( i_1 < i_2 < \ldots < i_k \) and there is a permutation \( \sigma \) of \( m = 1, \ldots, k \) such that \( j_{\sigma(m)} = i_m \) for \( m = 1, \ldots, k \). Again, since \( \phi \) is anti-symmetric, we have

\[
\phi(e_{j_1}, \ldots, e_{j_k}) = \text{sgn}(\sigma) \phi(e_{i_1}, \ldots, e_{i_k}).
\]

where \( \text{sgn}(\sigma) \) denotes the sign of the permutation \( \sigma \) (see the discussion of the sign of a permutation in section 17.1). Thus, from eqn (1), we get

\[
\phi(\bar{x}^1, \ldots, \bar{x}^k) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \sum_{\sigma} \text{sgn}(\sigma) x_{j_{\sigma(1)}}^1 \cdots x_{j_{\sigma(k)}}^k \phi(e_{i_1}, \ldots, e_{i_k})
\]

But by Leibniz formula for the determinant (as a sum of signed generalized diagonal products), this equals

\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \phi(e_{i_1}, \ldots, e_{i_k}) e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* (\bar{x}^1, \ldots, \bar{x}^k)
\]

and so

\[
\phi = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \phi(e_{i_1}, \ldots, e_{i_k}) e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*.
\]

**Linear Independence:** Suppose

\[
\sum_{i_1 < i_2 < \ldots < i_k} a_{i_1,i_2,\ldots,i_k} e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* = 0
\]

We must show that each \( a_{i_1,i_2,\ldots,i_k} = 0 \).

For each choice of \( \bar{x}^1, \ldots, \bar{x}^k \in V \), we have

\[
\sum_{i_1 < i_2 < \ldots < i_k} a_{i_1,i_2,\ldots,i_k} e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* (\bar{x}^1, \ldots, \bar{x}^k) = 0
\]

Given a fixed but arbitrary choice of \( j_1 < j_2 < \ldots < j_k \), choose \( \bar{x}^1 = e_{j_1}, \ldots, \bar{x}^k = e_{j_k} \).
Then

\[
\begin{bmatrix}
  x_{11}^1 & x_{12}^2 & \cdots & x_{1k}^k \\
x_{12}^1 & x_{12}^2 & \cdots & x_{12}^k \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
x_{1k}^1 & x_{1k}^2 & \cdots & x_{1k}^k \\
\end{bmatrix}
= \begin{bmatrix}
  (e_{j_1})_{i_1} & (e_{j_2})_{i_1} & \cdots & (e_{j_k})_{i_1} \\
  (e_{j_1})_{i_2} & (e_{j_2})_{i_2} & \cdots & (e_{j_k})_{i_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  (e_{j_1})_{i_k} & (e_{j_2})_{i_k} & \cdots & (e_{j_k})_{i_k} \\
\end{bmatrix}
\]  

(3)

If \((i_1, i_2, \ldots, i_k) = (j_1, j_2, \ldots, j_k)\) then eqn (3) is 1. If \((i_1, i_2, \ldots, i_k) \neq (j_1, j_2, \ldots, j_k)\), then for the smallest \(\ell\) s.t. \(i_\ell \neq j_\ell\), then either row \(\ell\) or column \(\ell\) of the matrix in eqn (3) is all zeros and so eqn (3) is 0.

Thus, with the choice, \(\pi^1 = e_{j_1}, \ldots, \pi^k = e_{j_k}\), eqn (2) becomes \(a_{j_1,j_2,\ldots,j_k} = 0\). Since the choice of \(j_1 < j_2 < \ldots < j_k\) was arbitrary, we get that each \(a_{j_1,j_2,\ldots,j_k} = 0\). Thus, we have linear independence.