Math 227:

**Major results of classical vector calculus:**

1. **Fundamental Theorem of Line Integrals:**
   
   (a) On a connected domain $D$, for a continuous vector field $\mathbf{F}$, the following are equivalent.
   
   - $\mathbf{F}$ is conservative (i.e., there exists a function $\phi$, called a potential function, such that $\mathbf{F} = \nabla \phi$).
   - $\mathbf{F}$ has path independent line integrals.
   - $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all simple closed curves $C$ in $D$.

   (b) On a simply connected domain, for a $C^1$ vector field $\mathbf{F}$, the three conditions above are also equivalent to: $\nabla \times (\mathbf{F}) = 0$.

2. **Green’s theorem:** Let $D$ be a closed, bounded region in $\mathbb{R}^2$ whose boundary $C = \partial D$ consists of a finite number of positively oriented, piecewise smooth, simple closed curves. Let $\mathbf{F}$ be a $C^1$ vector field on $D$. Then

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx \, dy$$

   equivalently,

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_D (\nabla \times \mathbf{F}) \cdot k \, dx \, dy$$

3. **Stokes Theorem:** Let $S$ be a piecewise smooth, oriented surface in $\mathbb{R}^3$ with surface boundary $C$ consisting of finitely many piecewise smooth simple closed curves with orientation consistent with the unit normal field $\mathbf{N}$ to $S$. Then

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{N} \, dS$$

   where $\mathbf{N}$ is the unit normal that orients $S$.

4. **Gauss (Divergence) Theorem:** Let $V$ be a bounded closed domain in $\mathbb{R}^3$ with boundary $\mathcal{S}$ consisting of finitely many piecewise smooth closed surfaces with outward unit normal field $\mathbf{N}$. Let $\mathbf{F}$ be a $C^1$ vector field on $V$. Then

   $$\int \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int \int_V (\nabla \cdot \mathbf{F}) \, dV$$

   where $\mathcal{S}$ is the 3-D boundary of $V$, with outward unit normal field on $\mathcal{S}$.
How to compute vector line integrals $\int_{C} \mathbf{F} \cdot d\mathbf{r}$

1. If $\mathbf{F}$ and $C$ are really nice, use the definition.

2. If $\mathbf{F}$ is conservative (for example, if $\nabla \times \mathbf{F} = \mathbf{0}$ and the domain is simply connected) and you know a potential $\phi$, try fundamental theorem of line integrals:

   $$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int \nabla \phi \cdot d\mathbf{r} = \phi(Q) - \phi(P)$$

   where $P$ is initial point and $Q$ is terminal point of $C$.

3. If $C$ is a simple closed curve and is the surface boundary of a nice surface $S$ and $\mathbf{F}$ is $C^1$ on $S$, try Stokes (if $C$ is planar, this reduces to Green):

   $$\int_{C=\partial S} \mathbf{F} \cdot d\mathbf{r} = \int \int_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

4. If there is another curve $C'$ s.t. $C + C'$ is a simple closed curve and is the surface boundary of a nice surface $S$ and $\mathbf{F}$ is $C^1$ on $S$, try Stokes:

   $$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int \int_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} - \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$
How to compute vector surface integrals (flux integrals) $\int_S \mathbf{F} \cdot d\mathbf{S}$:

1. If $\mathbf{F}$ and $S$ are really nice, use the definition.

2. If $S$ is a closed surface and bounds a nice solid region $V$ in $\mathbb{R}^3$ and $\mathbf{F}$ is $C^1$ on $V$, try Gauss:

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int \int_V \nabla \cdot \mathbf{F} \, dxdydz$$

3. If there is another surface $S'$ such that $S + S'$ bounds a nice solid region $V$ in $\mathbb{R}^3$ and $\mathbf{F}$ is $C^1$ on $V$, try Gauss:

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int \int_V \nabla \cdot \mathbf{F} \, dxdydz - \int \int_{S'} \mathbf{F} \cdot d\mathbf{S}$$

4. If $\mathbf{F}$ is a curl field (i.e., $\mathbf{F} = \nabla \times \mathbf{G}$ for some $\mathbf{G}$, for example if the domain is bounding and $\nabla \cdot \mathbf{F} = 0$), then

   (a) If $\int_{\partial S} \mathbf{G} \cdot d\mathbf{r}$ is easy to compute, apply Stokes:

   $$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{G} \cdot d\mathbf{r}$$

   (b) If there is a surface $S'$ such that $\partial S = \partial S'$ and $\int \int_S \mathbf{F} \cdot d\mathbf{S}$ is easy to compute, try Stokes:

   $$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} = \int_{\partial S = \partial S'} \mathbf{G} \cdot d\mathbf{r} = \int \int_{S'} \mathbf{F} \cdot d\mathbf{S}$$

   Note: this also follows from Gauss: $\nabla \cdot \mathbf{F} = 0$. So, for the solid $B$ bounded by $S$ and $S'$, we have

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} - \int \int_{S'} \mathbf{F} \cdot d\mathbf{S} = \int \int_B \nabla \cdot \mathbf{F} \, dV = 0.$$