Bonus Lecture (integration on manifolds in $\mathbb{R}^n$):

Defn: A **smooth parameterized $k$-manifold in** $\mathbb{R}^n$ is a continuous function

$$\bar{r} : D \to \mathbb{R}^n$$

where $D$ is a $k$-dimensional rectangular parallelepiped

$$D = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$$

such that on the interior of $D$

1. $\bar{r}$ is 1-1
2. $\bar{r}$ is $C^1$
3. $\{\frac{\partial \bar{r}}{\partial u_i}, i = 1, \ldots, k\}$ are linearly independent.

where

$$\frac{\partial \bar{r}}{\partial u_i} = \frac{d}{dt} \bar{r}(u_1, \ldots, u_{i-1}, t, u_i, \ldots, u_k)$$

which is the tangent vector to the $u_i$-curve.

$D$ can be generalized to other domains in $\mathbb{R}^k$.

$k = 1$ is a curve, $k = 2$ is a surface in $\mathbb{R}^n$. In particular, this generalizes the case $k = 2, n = 3$, which is a smooth parameterized surface with $\frac{\partial \bar{r}}{\partial u}, \frac{\partial \bar{r}}{\partial v}$ as tangents to the $u$-curves and $v$-curves and the linear independence condition is equivalent to

$$\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \neq 0.$$

Given a smooth parameterized $k$-manifold $\bar{r}$, with $\mathcal{M} := \bar{r}(D)$ and a differential $k$-form $\Phi \in \mathcal{F}_k(\mathcal{M})$

$$\int_\mathcal{M} \Phi := \int \cdots \int_D \Phi\left(\frac{\partial \bar{r}}{\partial u_1}, \ldots, \frac{\partial \bar{r}}{\partial u_k}\right) du_1 \cdots du_k$$
The integral is independent of the parameterization \( \tau \) of \( M \), provided that they induce the same orientation (to be defined below).

Example: \( k = 2, n = 3 \): so, \( M \) is a surface.

\[
\Phi = G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy
\]

\[
\int_M \Phi = \int \int_D G_1(\tau(u, v)) \left| \begin{array}{cc}
\left( \frac{\partial \tau}{\partial u} \right)_2 & \left( \frac{\partial \tau}{\partial v} \right)_2 \\
\left( \frac{\partial \tau}{\partial u} \right)_3 & \left( \frac{\partial \tau}{\partial v} \right)_3
\end{array} \right| dudv
\]

\[
+ \int \int_D G_2(\tau(u, v)) \left| \begin{array}{cc}
\left( \frac{\partial \tau}{\partial u} \right)_3 & \left( \frac{\partial \tau}{\partial v} \right)_3 \\
\left( \frac{\partial \tau}{\partial u} \right)_1 & \left( \frac{\partial \tau}{\partial v} \right)_1
\end{array} \right| dudv
\]

\[
+ \int \int_D G_3(\tau(u, v)) \left| \begin{array}{cc}
\left( \frac{\partial \tau}{\partial u} \right)_1 & \left( \frac{\partial \tau}{\partial v} \right)_1 \\
\left( \frac{\partial \tau}{\partial u} \right)_2 & \left( \frac{\partial \tau}{\partial v} \right)_2
\end{array} \right| dudv
\]

\[
= \int \int_D \overline{G} \cdot \left( \frac{\partial \tau}{\partial u} \times \frac{\partial \tau}{\partial v} \right) dudv
\]

\[
= \int \int_M \overline{G} \cdot dS
\]

where \( \overline{G} = (G_1, G_2, G_3) \).

This is a vector surface integral or flux integral (using the normal induced by \( \tau \)).

Example: \( k = 1, n = 3 \):

\[
\Phi = F_1 dx + F_2 dy + F_3 dz \quad \text{and} \quad M \text{ is a parameterized 1-manifold.}
\]

Then

\[
\int_M \Phi = \int_M \overline{F} \cdot d\overline{\tau}
\]

where \( \overline{F} = (F_1, F_2, F_3) \).

Similar to the above, you can show that this is a vector line integral (using the tangent induced by \( \tau \)).

General Stokes Theorem: Let \( M \) be an oriented \( k \)-manifold in \( \mathbb{R}^n \) with or without boundary \( \partial M \), with orientation consistent with
orientation of \( \mathcal{M} \). Let \( \Phi \in \mathcal{F}_{k-1}(\mathcal{M}) \), a differential \((k - 1)\)-form on \( \mathcal{M} \). Then

\[
\int_{\mathcal{M}} d\Phi = \int_{\partial \mathcal{M}} \Phi
\]

Here, \( \partial \mathcal{M} \) denotes the “manifold boundary” not the \( \mathbb{R}^n \) boundary.

This generalizes fundamental theorem of calculus, fundamental theorem of line integrals, Green, Gauss and Stokes.

Example: For \( k = 2, n = 3 \), General Stokes reduces to ordinary Stokes because if we identify \( \Phi = F_1dx + F_2dy + F_3dz \) with \( \overline{F} = (F_1, F_2, F_3) \), then \( d\Phi \) is identified with the curl, \( \nabla \times \overline{F} \).

Defn: A \( k \)-manifold in \( \mathbb{R}^n \) is a closed subset of \( \mathbb{R}^n \) such that for each \( \overline{p} \in \mathcal{M} \), there is an \( n \)-dimensional ball \( B \) centered at \( \overline{p} \) such that \( \mathcal{M} \cap B \) is the image of the interior of a parametrized \( k \)-manifold.

Such a parameterization is viewed as a “local parameterization.”

“Defn”: A \( k \)-manifold with boundary in \( \mathbb{R}^n \) is the same as a \( k \)-manifold except for some “lower dimensional boundary pieces” which include only finitely many \((k-1)\)-dimensional manifolds, whose union is denoted \( \partial \mathcal{M} \).

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Note: you can also consider piecewise smooth manifolds and define orientations on them and their manifold boundaries.

Orientation of Manifolds and their manifold boundaries

Defn: An orientation on a \( k \)-manifold \( \mathcal{M} \) is an equivalence class of non-zero \( k \)-forms \( \omega \) on \( \mathcal{M} \) (called orientation forms).
Here, **non-zero** means that for any choice of linearly independent continuous vector fields $\vec{v}^1, \ldots, \vec{v}^k$ tangent to $\mathcal{M}$, for all $\vec{p} \in \mathcal{M}$, $\omega(\vec{p})(\vec{v}^1, \ldots, \vec{v}^k) \neq 0$. And two such forms $\omega_1, \omega_2$ are **equivalent** if there is a *positive* continuous function $f$ s.t. $\omega_2 = f \omega_1$.

Here, a vector is tangent to $\mathcal{M}$ at $\vec{p}$ if it belongs to the tangent space of $\mathcal{M}$ at $\vec{p}$ which is the linear span of tangents to the $u_1, u_2, \ldots, u_k$ curves at $\vec{p}$, given by a local parameterization.

An orientation form is used only to determine the signs needed to determine consistency of orientation between $\mathcal{M}$ and $\partial \mathcal{M}$.

Example: A 2-manifold $\mathcal{M}$ in $\mathbb{R}^3$:

$$\omega(\vec{u}, \vec{v}) = det(\overrightarrow{N}, \vec{u}, \vec{v}) = \overrightarrow{N} \cdot (\vec{u} \times \vec{v})$$

where $\vec{u}, \vec{v}$ are vector fields on $\mathcal{M}$ and $\overrightarrow{N}$ is a continuous nonzero normal vector field on $\mathcal{M}$.

This generalizes to all $n$ by finding a basis for the tangent space at each point given a smooth parameterization

$$\left\{ \frac{\partial \mathcal{T}}{\partial u} = (a_1, \ldots, a_n), \frac{\partial \mathcal{T}}{\partial v} = (b_1, \ldots, b_n) \right\}$$

and defining

$$\omega = \left( \sum_{i=1}^{n} a_i dx_i \right) \wedge \left( \sum_{i=1}^{n} b_i dx_i \right)$$

Example: A 1-manifold $\mathcal{M}$ in $\mathbb{R}^3$:

$$\omega(\vec{u}) = \vec{u} \cdot \mathcal{T}$$

where $\vec{u}$ is a vector field on $\mathcal{M}$ and $\mathcal{T}$ is a continuous nonzero tangent vector field on $\mathcal{M}$.

This generalizes from $n = 3$ to all $n$. 
Note that for $n = 2$, $\overline{u} \cdot \overline{T} = \det (\overline{N}, \overline{u})$ where $\overline{N} = (-T_2, T_1)$.

Defn: Given a $k$-manifold with manifold boundary $\partial \mathcal{M}$ and orientation $k$-form $\omega$ on $\mathcal{M}$, the induced orientation $(k - 1)$-form $\partial \omega$ on $\partial \mathcal{M}$ is defined by

$$\partial \omega(\overline{v}^1, \ldots, \overline{v}^{k-1}) = \omega(\overline{Z}, \overline{v}^1, \ldots, \overline{v}^{k-1})$$

where

1. $\overline{v}_1, \ldots, \overline{v}_{k-1}$ and $\overline{Z}$ are vector fields on $\partial \mathcal{M}$
2. $\overline{Z}$ is tangent to $\mathcal{M}$
3. $\overline{Z}$ is normal to $\partial \mathcal{M}$.
4. $\overline{Z}$ points away from $\mathcal{M}$.

Note that $\overline{Z}$ is unique up to positive scalar multiples.

Example: $k = 2$ and $n = 3$

An oriented 2-manifold $\mathcal{S}$ is an oriented surface in $\mathbb{R}^3$. If $\mathcal{S}$ is a closed surface, then $\partial \mathcal{S} = \emptyset$. Otherwise, $\partial \mathcal{S}$ consists of finitely many consistently oriented, piecewise smooth, simple closed curves. As above,

$$\omega(\overline{u}, \overline{v}) = det(\overline{N}, \overline{u}, \overline{v})$$

$$\partial \omega(\overline{u}) = \omega(\overline{Z}, \overline{u}) = det(\overline{N}, \overline{Z}, \overline{u}) = \overline{u} \cdot (\overline{N} \times \overline{Z})$$

Note that $\overline{T} = \overline{N} \times \overline{Z}$ is tangent to $\partial \mathcal{S}$ and points in the direction such that $\mathcal{S}$ is to the left of $\partial \mathcal{S}$. 

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Example: $k = 1$ and $n = 3$.

An oriented 1-manifold $C$ is an oriented curve in $\mathbb{R}^n$, say a simple curve (for simplicity, please excuse the pun). If $C$ is a simple closed curve, then $\partial C = \emptyset$. Otherwise, $\partial C = \{\overline{P}, \overline{Q}\}$, two distinct points, the initial point $\overline{P}$ and terminal point $\overline{Q}$.

An orientation form on $C$ is given by $\omega(p)(v) = v \cdot T(p)$ where $T$ is a non-zero continuous tangent vector field along $C$.

So $\partial_\omega(\overline{P}) = \overrightarrow{Z} \cdot T(\overline{P}) < 0$ and $\partial_\omega(\overline{Q}) = \overrightarrow{Z} \cdot T(\overline{Q}) > 0$.

So, for a differential 0-form $\Phi = f$, a $C^\infty$ function, with parameterization $\overline{r}$ and $\overline{r}(a) = \overline{P}$, $\overline{r}(b) = \overline{Q}$, we have

$$\int_C \nabla f \cdot d\overline{r} = \int_C d\Phi = \int_{\partial C} \Phi = \int_{\{\overline{P}, \overline{Q}\}} \Phi = f(\overline{Q}) - f(\overline{P}).$$

since when $\Phi = f$ is a function $d\Phi = df$ corresponds to $\nabla f$. 