JUSTIFY ALL OF YOUR ANSWERS. YOU MAY USE RESULTS FROM CLASS AND HOMEWORK.
CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.
THERE ARE 4 PROBLEMS ON THIS EXAM.
1. Let $\bar{r}(t) = ((1/3)(1 + 2t)^{3/2}, (1/2)t^2), \ 0 \leq t \leq 1$.

   (a) Find the arclength parameterization of the oriented curve which is the image of $\bar{r}(t)$.

   (b) Find the arclength parameterization of the same curve but with reversed orientation.
Solution:

a. \( \vec{r}'(t) = (\sqrt{1 + 2t}, t) \) and so
\[
||\vec{r}'(t)|| = \sqrt{1 + 2t + t^2} = 1 + t
\]
So
\[
s = s(t) = \int_0^t (1 + u)du = t + (1/2)t^2
\]
So \( t = t(s) \) is determined by the quadratic equation: \((1/2)t^2 + t - s = 0\). Solving for \( t \), we get
\[
t = -1 \pm \sqrt{1 + 2s}
\]
The minus solution does not work since \( t \geq 0 \). Thus,
\[
t = t(s) = -1 + \sqrt{1 + 2s}
\]
and the arclength parameterization is
\[
\vec{x}(s) = ((1/3)(-1 + 2\sqrt{1 + 2s})^{3/2}, (1/2)(-1 + \sqrt{1 + 2s})^2),
\]

b. The length of the curve is \( s(1) = 3/2 \). So,
\[
\vec{y}(s) = \vec{x}(3/2 - s) = ((1/3)(-1+2\sqrt{4 - 2s})^{3/2}, (1/2)(-1+\sqrt{4 - 2s})^2),
\]
\[0 \leq s \leq 3/2.\]
An alternative is to reverse the original parameterization of \( \vec{r}(t) \) and then find the arclength parameterization from that. But that involves more work.
2. Let $\mathbf{r}(t)$ be a $C^2$ smooth parameterization, $\mathbf{v}(t) := \mathbf{r}'(t)$, and $\mathbf{a}(t) := \mathbf{r}''(t)$. Fix $t$.

(a) Show that $\mathbf{a}(t)$ is orthogonal to $\mathbf{v}(t)$ iff $||\mathbf{v}(t)||' = 0$.

(b) Show that $\mathbf{a}(t)$ is parallel to $\mathbf{v}(t)$ iff $\kappa(t) = 0$. 
Solution:

a. We know from class that for any $C^1$ parameterization $\vec{x}(t)$, $||\vec{x}(t)||$ is constant iff $\vec{x}(t) \cdot \vec{x}'(t) = 0$. Apply this to $\vec{x}(t) := \vec{v}(t)$ and we get $||\vec{v}(t)||' = 0$ iff $||\vec{v}(t)||$ is constant iff $\vec{v}(t) \cdot \vec{a}(t) = 0$.

b. $\vec{a}(t)$ is parallel to $\vec{v}(t)$ iff $\vec{v}(t) \times \vec{a}(t) = \vec{0}$. This latter condition is equivalent to $\kappa(t) = 0$ by the formula for curvature:

$$
\kappa(t) = \frac{||\vec{v}(t) \times \vec{a}(t)||}{||\vec{v}(t)||^3}
$$

Alternatively, one can use the formula:

$$
\vec{a}(t) = ||\vec{r}'(t)||'\vec{T}(t) + ||\vec{r}'(t)||^2\kappa(t)\vec{N}(t).
$$
3. Consider a $C^4$ smooth curve with nowhere-zero curvature. Let

$$\overrightarrow{w}(s) = \tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s)$$

(a) Show that $||\overrightarrow{w}(s)|| = \sqrt{(\kappa(s))^2 + (\tau(s))^2}$.

(b) Show that $\mathbf{N}'(s) = \overrightarrow{w}(s) \times \mathbf{N}(s)$

(c) Show that the curve is a helix iff $\overrightarrow{w}(s)$ is a constant vector.
Solution:

a. \[ \|\overline{w}(s)\| = \sqrt{(\tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s)) \cdot (\tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s))} \]
\[ = \sqrt{(\tau(s))^2\mathbf{T}(s) \cdot \mathbf{T}(s) + (\kappa(s))^2\mathbf{B}(s) \cdot \mathbf{B}(s) + 2\tau(s)\kappa(s)\mathbf{T}(s) \cdot \mathbf{B}(s)} \]
\[ = \sqrt{(\kappa(s))^2 + (\tau(s))^2} \]

since \( \mathbf{T}(s) \) and \( \mathbf{B}(s) \) are orthogonal unit vectors.

b. \[ \overline{w}(s) \times \mathbf{N}(s) = (\tau(s)\mathbf{T}(s) + \kappa(s)\mathbf{B}(s)) \times \mathbf{N}(s) \]
\[ = \tau(s)\mathbf{T}(s) \times \mathbf{N}(s) + \kappa(s)\mathbf{B}(s) \times \mathbf{N}(s) = \tau(s)\mathbf{B}(s) - \kappa(s)\mathbf{T}(s) \]

From the Frenet formulas, we get
\[ \mathbf{N}'(s) = \tau(s)\mathbf{B}(s) - \kappa(s)\mathbf{N}(s) \]

. 

c. Note that \( \tau \) and \( \kappa \) and the frame vectors are differentiable by our \( C^4 \) assumption, So, \( \overline{w}(s) \) is constant iff \( \overline{w}'(s) = 0 \).

\[ \overline{w}'(s) = \tau'(s)\mathbf{T}(s) + \tau(s)\mathbf{T}'(s) + \kappa'(s)\mathbf{B}(s) + \kappa(s)\mathbf{B}'(s) \]
\[ = \tau'(s)\mathbf{T}(s) + \tau(s)\kappa(s)\mathbf{N}(s) + \kappa'(s)\mathbf{B}(s) - \kappa(s)\tau(s)\mathbf{N}(s) \]
\[ = \tau'(s)\mathbf{T}(s) + \kappa'(s)\mathbf{B}(s) \]

By linear independence of \( \mathbf{T} \) and \( \mathbf{B} \), \( \overline{w}' = \overline{0} \) iff \( \tau' = 0 \) and \( \kappa' = 0 \), which is equivalent to \( \kappa \) and \( \tau \) being constant. But we know from homework (using the fundamental theorem of space curves), that constancy of \( \kappa \) and \( \tau \) is equivalent to the original curve being a helix (the constant curvature must be positive by our assumption on \( C \); if the constant torsion is zero, then the curve is a circle, which is a degenerate helix).
4. Let \( f(t) = t^4 \sin(1/t) \) if \( t \neq 0 \) and \( f(0) = 0 \). Let

\[
\vec{r}(t) = \begin{cases} 
(t, f(t), 0) & t \geq 0 \\
(t, 0, f(t)) & t < 0
\end{cases}
\]

(a) Show that \( \vec{r}(t) \) is a smooth parameterization.
(b) Find the unit tangent vector at \( t = 0 \).
(c) Find the curvature at \( t = 0 \).
(d) Show that for each \( t \), either the curvature is 0 or the torsion is 0.
Solution:

a. For \( t \neq 0 \),

\[
f'(t) = 4t^3 \sin(1/t) - t^2 \cos(1/t)
\]

and

\[
f'(0) = \lim_{t \to 0} \frac{t^4 \sin(1/t)}{t} = \lim_{t \to 0} t^3 \sin(1/t) = 0.
\]

So, \( f \) is differentiable. Clearly \( f(t) \) is \( C^1 \) for all \( t \neq 0 \). Now,

\[
\lim_{t \to 0} f'(t) = \lim_{t \to 0} 4t^3 \sin(1/t) - t^2 \cos(1/t) = 0
\]

and so \( f \) is \( C^1 \) at \( t = 0 \) (note that we proved \( f'(0) = 0 \) above separately from \( \lim_{t \to 0} f'(t) = 0 \). But it turns out that the latter + continuity of \( f \) at 0 proves the former).

It also follows that the functions

\[
g(t) = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}
\]

and

\[
h(t) = \begin{cases} 0 & t \geq 0 \\ f(t) & t < 0 \end{cases}
\]

are \( C^1 \). Since \( \vec{r}(t) = (t, g(t), h(t)) \), it is \( C^1 \) and \( \vec{r}'(t) = (1, g'(t), h'(t)) \) is never 0. Thus, \( \vec{r}(t) \) is smooth.

b. Since \( f'(0) = 0 \),

\[
\vec{r}'(t) = (1, 0, 0)
\]

and so

\[
\mathbf{T}(t) = (1, 0, 0)
\]

c.

\[
f''(0) = \lim_{t \to 0} \frac{4t^3 \sin(1/t) - t^2 \cos(1/t)}{t} = \lim_{t \to 0} 4t^2 \sin(1/t) - t \cos(1/t) = 0.
\]
Thus, $g'(0) = g''(0) = h'(0) = h''(0) = 0$, and so
\[
\kappa(0) = \frac{|| (1, g'(0), h'(0)) \times (0, g''(0), h''(0)) ||}{|| (1, g'(0), h'(0)) ||^3} = \frac{||(1, 0, 0) \times (0, 0, 0)||}{1} = 0.
\]

d. First observe that for $t \neq 0$,
\[
f''(t) = 12t^2 \sin(1/t) - 4t \cos(1/t) - 2t \cos(1/t) + \sin(1/t)
\]
and thus $f''$ is continuous in some nbhd. of any $t_0 \neq 0$.

By part c, $\kappa(0) = 0$. Suppose for some $t_0$, $\kappa(t_0) > 0$. Necessarily, $t_0 \neq 0$. Since,
\[
\kappa(t) = \frac{|| (1, g'(t), h'(t)) \times (0, g''(t), h''(t)) ||}{|| (1, g'(t), h'(t)) ||^3}
\]
and $f''$ is continuous in a nbhd. of $t_0$, $\kappa(t) > 0$ in some nbhd. of $t_0$. Thus, the Frenet frame, in particular $B(t)$, is defined in this nbhd. But since restricted to either $(0, \infty)$ or $(-\infty, 0)$, the image of $\vec{r}$ is planar, $B(t)$ is constant in that nbhd. and thus its derivative is 0 in that nbhd. Thus, by definition of torsion, $\tau(t_0) = 0$.

Note that $f$ is not $C^2$ at $t = 0$. 