1. Let \( f(x, y) = x^4 + y^4 - 4xy + 6 \).
   
   (a) Find and classify all critical points of \( f \).
   
   (b) Determine if \( f \) has a global maximum and global minimum and find them if they exist. Explain your answer.

   Solution: Just like Problem 4, Section 13.1. The justification given there for the existence of the global minimum is somewhat technical. The main idea is just that \( \lim_{(x, y) \to \infty} f(x, y) = +\infty \).

2. Suppose that \( T(x, y, z) = 3 + xy - y^2 + z^2 - x \) describes the temperature at any point \((x, y, z)\) in space.
   
   (a) At the point \( (3, 2, 1) \), in what direction does the temperature decrease most rapidly?
   
   (b) Find the directional derivative of \( T \) at \( (3, 2, 1) \) in the direction \((0, 1, 2)\).
   
   (c) Suppose that for some value \( b \), the vector \((1, 5, b)\) is tangent to the level surface of \( T \) at level \( (3, 2, 1) \). What is \( b \)?

   Solution: Typo: In part c, the last occurrence of the word “level” should not be there.

   Part a.
   
   \[ \nabla T = (y - 1, x - 2y, 2z) \]
   
   At the point \( (3, 2, 1) \),
   
   \[ \nabla T = (1, -1, 2) \]
   
   Thus, the temperature is decreasing most rapidly in direction \( (-1, 1, -2)/\sqrt{6} \).

   Part b.

   Since \( T \) is a polynomial, it is differentiable and so

   \[ D_\pi T = \nabla T \cdot \pi = (1, -1, 2) \cdot (0, 1, 2)/\sqrt{5} = 3/\sqrt{5}. \]

   Part c.

   \[ (1, 5, b) \cdot (1, -1, 2) = 0 \]
   
   So, \(-4 + 2b = 0\). So, \( b = 2 \).
3. The functions \( x(r, s) \) and \( y(r, s) \) are defined implicitly by the equations
\[
e^x - e^y = r, \quad 2x + y = s
\]
Find \( x(0, 3), y(0, 3), \frac{\partial x}{\partial r}(0, 3), \frac{\partial y}{\partial r}(0, 3), \) and \( \frac{\partial^2 x}{\partial r^2}(0, 3), \)
Solution: To find \( x(0, 3) \) and \( y(0, 3), \)
\[
e^x - e^y = 0, \quad 2x + y = 3.
\]
Thus, \( x(0, 3) = y(0, 3) = 1. \)
\[
\frac{\partial x}{\partial r} = -\frac{\partial (F,G)}{\partial (r,y)}
\]
Here, \( F(x, y, r, s) = e^x - e^y - r, \) \( G(x, y, r, s) = 2x + y - s, \) and the numerator is
\[
\begin{vmatrix}
\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\
\frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}
\end{vmatrix} = \begin{vmatrix} -1 & -e^{y(r,s)} \\ 0 & 1 \end{vmatrix} = -1
\]
and the denominator is
\[
\begin{vmatrix}
e^{x(r,s)} & -e^{y(r,s)} \\
2 & 1
\end{vmatrix} = e^{x(r,s)} + 2e^{y(r,s)}
\]
Thus,
\[
\frac{\partial x}{\partial r} = (e^{x(r,s)} + 2e^{y(r,s)})^{-1}
\]
A similar computation yields
\[
\frac{\partial y}{\partial r} = -2(e^{x(r,s)} + 2e^{y(r,s)})^{-1}
\]
So,
\[
\frac{\partial x}{\partial r}(0, 3) = \frac{1}{3e}, \quad \frac{\partial y}{\partial r}(0, 3) = \frac{-2}{3e}.
\]
Finally,
\[
\frac{\partial^2 x}{\partial r^2} = -(e^{x(r,s)} + 2e^{y(r,s)})^{-2}(e^{x(r,s)}\frac{\partial x}{\partial r} + 2e^{y(r,s)}\frac{\partial y}{\partial r})
\]
So,
\[
\frac{\partial^2 x}{\partial r^2}(0, 3) = -(3e)^{-2}((-\frac{1}{3e}e - \frac{4}{3e}e) = \frac{1}{9e^2}
\]
Note: this problem can also be solved more directly using the chain rule.

4. Let
\[
f(x, y) = \begin{cases} 
\frac{x^5}{x+y^4} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]
(a) Find a general formula for the directional derivative of $f$ at $(0,0)$ for every unit vector $(u_1, u_2)$.

(b) Is $f$ differentiable at $(0,0)$? Why or why not?

(c) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function and $D_\vec{u} f(0,0) = u_1^2 - u_2^2$ for every unit vector $\vec{u} = (u_1, u_2)$. Show that $f$ can NOT be differentiable at $(0,0)$?

Solution:

Part a.

$$D_\vec{u} f(0,0) = \lim_{h \to 0} \frac{f(h\vec{u}) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^5 u_1^5}{h(h^4 u_1^4 + h^4 u_2^4)} = \frac{u_1^5}{u_1^4 + u_2^4}$$

Part b.

It follows from part a that $\nabla f(0,0) = (1, 0)$.

If $f$ were differentiable at $(1,0)$, then

$$D_\vec{u} f(0,0) = (\nabla f(0,0)) \cdot (u_1, u_2) = u_1.$$ 

Then, by part a, we would get, for all unit vectors:

$$\frac{u_1^5}{u_1^4 + u_2^4} = u_1$$

But this is false – for example, it doesn’t hold when $u_2 \neq 0$.

Thus, $f$ is not differentiable at $(0,0)$.

Part c. No.

Suppose that $f$ is differentiable at $(0,0)$, then again

$$D_\vec{u} f(0,0) = (\nabla f(0,0)) \cdot (u_1, u_2) = au_1 + bu_2$$

for some numbers $a$ and $b$ (the partial derivatives).

For $\vec{u} = (1,0)$: we get $D_\vec{u} f(0,0) = a$ and $u_1^2 - u_2^2 = 1$. So, $a = 1$.

For $\vec{u} = (-1,0)$: we get $D_\vec{u} f(0,0) = -a$ and $u_1^2 - u_2^2 = 1$. So, $a = -1$.

This contradiction shows that $f$ is not differentiable at $(0,0)$.