Math 226, Sample Midterm 2 Solution to Problem 1

1. Let \( f(x, y) = x^4 + y^4 - 4xy + 6 \).
   a. Find and classify all critical points of \( f \).
   
   Solution:
   
   a. Find critical points:

   \[
   f_x = 4x^3 - 4y, \quad f_y = 4y^3 - 4x
   \]

   Setting \( f_x = 0, f_y = 0 \), we find \( y = x^3, x = y^3 \) and so \( x = x^9 \). Then only real solutions to this equation are \( x = 0, 1, -1 \). Since \( y = x^3 \), we find that \( y = x \) and so the critical points are \( (0, 0), (1, 1), (-1, -1) \).

   \[
   f_{xx} = 12x^2, \quad f_{xy} = f_{yx} = -4, \quad f_{yy} = 12y^3.
   \]

   Note that \( f_{xy} = f_{yx} \) since \( f \) is \( C^2 \).
   
   For \((0, 0)\), \( A = C = 0, B = -4 \), so \( B^2 - AC = 16 > 0 \), and so this is a saddle.
   
   For \((1, 1)\) and \((-1, -1)\), \( A = C = 12, B = -4 \), so \( B^2 - AC = 16 - 144 < 0 \), and so these are relative min.

   b. Since \( f(x, 0) = x^4 + 6 \), there is no global max.
   
   Since degree four polynomials dominate degree two polynomials, \( x^4 \) and \( y^4 \) dominate \( xy \) and so

   \[
   \lim_{(x,y) \to \infty} f(x, y) = +\infty.
   \]

   So, for sufficiently large \( R \), all points outside the open disk \( B_R(\bar{0}) \) centered at \( \bar{0} \) with radius \( R \), cannot compete for the global minimum. So, if there is a global minimum, then it must occur in \( B_R(\bar{0}) \) and therefore at the relative minimum above. These are global minima because \( f(x, y) \) is a continuous function and the closed disk \( \overline{B_R(\bar{0})} \) is closed and bounded.