JUSTIFY ALL OF YOUR ANSWERS.
CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.
THERE ARE 4 PROBLEMS ON THIS EXAM.

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[25 marks]
Let \( f(x, y) = x^2 - 2xy + 5y^2 + 8y + 1 \).

(a) Find and classify all critical points of \( f \).

(b) Determine if \( f \) has a global maximum and/or a global minimum and, if so, find them. Explain your answer.

Solution:

Part a:
\[
\frac{\partial f}{\partial x} = 2x - 2y, \quad \frac{\partial f}{\partial y} = -2x + 10y + 8
\]
Solving the equations \( \frac{\partial f}{\partial x} = 0 \), \( \frac{\partial f}{\partial y} = 0 \) simultaneously, we find that the only critical point is \((-1, -1)\), with value \( f(-1, -1) = -3 \).

\[
\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2, \quad \frac{\partial^2 f}{\partial y^2} = 10
\]
So, \( A = 2 > 0 \) and \( B^2 - AC = -16 < 0 \). So, \((-1, -1)\) is a local minimum.

Part b: Since \( f(x, 0) = x^2+1 \) is unbounded, there is no global maximum.

By completing the square, we see that for all \((x, y)\),
\[
f(x, y) = (x - y)^2 + 4(y + 1)^2 - 3 \geq -3 = f(-1, -1)
\]
So, \((-1, -1)\) is a global minimum and the global minimum value is \(-3\). There are no other global minima since for \((x, y) \neq (-1, -1)\), we have \( f(x, y) > 3 \).

Note: In this solution, part a is not needed for part b.
Alternatively, one can argue first that \( \lim_{(x,y) \to \infty} f(x, y) = +\infty \) because 
\[ x^2 + y^2 \geq 2xy \] and so 
\[ f(x, y) \geq 4y^2 + 8y + 1 \to +\infty. \]
Then use the fact that for sufficiently large \( R \), letting \( D_R \) denote the closed disk of radius \( R \), \( \min f|_{D_R} \), which exists because of Theorem 2, is the same as \( \min f \), and must occur at the local minimum \( (-1, -1) \).
2. [25 marks]

Let $f(x, y)$ be a function that is differentiable at $(0, 0)$. Let $\overline{u}$ be the unit vector in direction $(-1, -1)$ and $\overline{v}$ be the unit vector in direction $(-1, 1)$. Suppose

$$D_{\overline{u}}f((0, 0)) = -1 \text{ and } D_{\overline{v}}f(0, 0) = 3.$$  

Let $g(t) = (h(t), k(t))$ be a differentiable path s.t. $h(0) = k(0) = 0$, $h'(0) = 1$ and $k'(0) = 4$. Let $F(t) = f(g(t))$. Find $F'(0)$.

**Solution:** Since $f$ is differentiable at $(0, 0)$, by the chain rule we have

$$F'(0) = f_x(g(0))h'(0) + f_y(g(0))k'(0) = f_x(0, 0) + 4f_y(0, 0)$$

Since $f$ is differentiable, at $(0, 0)$,

$$D_{\overline{u}}f = -(f_x)(1/\sqrt{2}) - (f_y)(1/\sqrt{2})$$

and

$$D_{\overline{v}}f = -(f_x)(1/\sqrt{2}) + (f_y)(1/\sqrt{2})$$

So, we have

$$f_x + f_y = \sqrt{2}, \quad f_y - f_x = 3\sqrt{2}.$$  

Solving simultaneously $f_y = 2\sqrt{2}$ and $f_x = -\sqrt{2}$. So, plugging in above, we have

$$F'(0) = -\sqrt{2} + 8\sqrt{2} = 7\sqrt{2}$$
3. [25 marks]

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \) function. For a fixed unit vector \( \bar{u} \in \mathbb{R}^2 \), view the directional derivative, \( D_{\bar{u}}f(\bar{x}) \), as a function of \( \bar{x} \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \). For a fixed unit vector \( \bar{v} \in \mathbb{R}^2 \), let \( D_{\bar{u}, \bar{v}}f(\bar{x}) \) be the directional derivative of the function \( D_{\bar{u}}f(\bar{x}) \) in direction \( \bar{v} \).

(a) Show that \( D_{\bar{u}, \bar{v}}f(\bar{x}) \) exists and in fact that
\[
D_{\bar{u}, \bar{v}}f(\bar{x}) = \bar{u} \mathcal{H} \bar{v}^T = \bar{v} \mathcal{H} \bar{u}^T
\]
where \( \mathcal{H} \) is the Hessian of \( f \) at \( \bar{x} \).

(b) Show that \( \mathcal{H} \) is positive definite iff for all unit vectors \( \bar{u} \),
\[
D_{\bar{u}, \bar{u}}f(\bar{x}) > 0.
\]

Solution:

a. Since \( f \) is \( C^2 \), it is differentiable. Thus,
\[
D_{\bar{u}}f(\bar{x}) = (\nabla f) \cdot \bar{u} = (f_x, f_y) \cdot (u_1, u_2) = u_1 f_x + u_2 f_y
\]
Thus, since \( f \) is \( C^2 \), \( f_x \) and \( f_y \) are \( C^1 \) and hence differentiable, and so
\[
D_{\bar{u}, \bar{v}}f = D_{\bar{v}}(D_{\bar{u}}f) = D_{\bar{v}}(u_1 f_x + u_2 f_y) = u_1 D_{\bar{v}} f_x + u_2 D_{\bar{v}} f_y
\]
\[
= u_1 (\nabla f_x) \cdot \bar{v} + u_2 (\nabla f_y) \cdot \bar{v} = u_1 (v_1 f_{xx} + v_2 f_{xy}) + u_2 (v_1 f_{yx} + v_2 f_{yy})
\]
This shows that \( D_{\bar{u}, \bar{v}}f \) exists and
\[
D_{\bar{u}, \bar{v}}f = [u_1, u_2] \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} [v_1, v_2]^T = \bar{u} \mathcal{H} \bar{v}^T
\]
Since \( f \) is \( C^2 \), \( \mathcal{H} \) is symmetric and so
\[
\bar{u} \mathcal{H} \bar{v}^T = (\bar{u} \mathcal{H} \bar{v}^T)^T = \bar{v} \mathcal{H} \bar{u}^T = \bar{v} \mathcal{H} \bar{u}^T
\]

b. “if:” Any vector \( \bar{w} \) that is not the zero vector, can be written \( \bar{w} = \lambda \bar{u} \) where \( \bar{u} \) is a unit vector and \( \lambda \neq 0 \). So, by part a,
\[
\bar{w} \mathcal{H} \bar{w}^T = \lambda^2 \bar{u} \mathcal{H} \bar{u}^T = \lambda^2 D_{\bar{u}, \bar{u}}f(\bar{x}) > 0,
\]
and so $\mathcal{H}$ is positive definite.

“only if.” If $\mathcal{H}$ is positive definite, then since any unit vector is not the zero vector,

$$D_{\overline{u},\overline{u}} f = \overline{u} \mathcal{H} \overline{u}^T > 0.$$
4. [25 marks]
For each of the following functions $f$, determine if $f$ is differentiable at $(0, 0)$. Explain your answers. You do NOT need to give $\epsilon - \delta$ proofs.

(a) 
$$f(x, y) = \frac{e^{\cos(x^2+y^3)}}{3 + 6x^2 + 7y^4}$$

(b) 
$$f(x, y) = \begin{cases} 
\frac{x \sin(2x^2+y^2)}{2x^2+y^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0) 
\end{cases}$$

You may use from 1-variable calculus the fact that \(\lim_{x \to 0} \frac{\sin(x)}{x} = 1\).

**Solution:**

a. By the chain rule, the partial derivatives of the numerator and denominator are compositions of polynomials, 1-variable exponentials and 1-variable trig functions, which are all continuous. So, the numerator and denominator have continuous partial derivatives and are therefore $C^1$. So, $f$ is a quotient of two $C^1$ functions with non-vanishing denominator. By the quotient rule (applied to partial derivatives) writing $f = g/h$, 

$$f_x = \frac{(h g_x - g h_x)}{h^2}, \quad f_y = \frac{(h g_y - g h_y)}{h^2},$$

which are continuous since both $g$ and $h$ are $C^1$ and $h$ is non-zero. Thus, $f$ is $C^1$ and hence $f$ is differentiable.

Alternatively, one can directly compute $f_x$ and $f_y$ and observe that they are quotients of compositions of continuous functions, with non-vanishing denominator, and so $f$ is $C^1$ and hence $f$ is differentiable.

Another alternative is to show that the quotient of two differentiable functions, with non-vanishing denominator, is differentiable but this is a more complicated argument, replicating the argument for the analogous result in one-variable calculus.
b. First compute the partial derivatives at \((0, 0)\).

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h \sin(2h^2)}{2h^3} = \lim_{h \to 0} \frac{\sin(2h^2)}{2h^2} = 1.
\]

Since \(f(0, k) = 0\) for all \(k\),

\[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, k) - f(0, 0)}{k} = 0.
\]

The question of differentiability then becomes

\[
\lim_{(x,y) \to (0,0)} \frac{x \sin(2x^2+y^2)}{(x^2 + y^2)^{1/2}} - x = 0.
\]

This equals

\[
\left( \frac{x}{(x^2 + y^2)^{1/2}} \right) \left( \frac{\sin(2x^2 + y^2)}{2x^2 + y^2} - 1 \right)
\]

The first factor is bounded and the second factor tends to 0. So, \(f\) is differentiable at \((0, 0)\).