CALCULATORS, NOTES OR BOOKS ARE NOT PERMITTED.
THERE ARE 4 PROBLEMS ON THIS EXAM.
JUSTIFY YOUR ANSWERS.

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[25 marks]

(a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Show that $|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2 |\mathbf{v}|^2$. Hint: use geometric definitions of $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$.

(b) Prove that for a parallelogram the sum of the squares of the lengths of the sides is the same as the sum of the squares of the lengths of the diagonals. Hint: use vectors and dot products.

Solution:

a. If either $\mathbf{u}$ or $\mathbf{v}$ is $\mathbf{0}$, the result is obvious. If not, let $\theta$ denote the angle in $[0, \pi]$ between $\mathbf{u}$ and $\mathbf{v}$. Then,

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin(\theta) \text{ and } \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta).$$

So,

$$|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 (\sin^2(\theta) + \cos^2(\theta)) = |\mathbf{u}|^2 |\mathbf{v}|^2.$$ 

Alternatively, a tedious calculation using the algebraic definitions will do the trick.

b. Let $P, Q, R, S$ be the vertices of the parallelogram in clockwise order. Let $\mathbf{u} := \overrightarrow{PQ}$ and $\mathbf{v} := \overrightarrow{PS}$. Then $\overrightarrow{QR} = \mathbf{v}$ and $\overrightarrow{SR} = \mathbf{u}$. So the sum of the squares of the lengths of the sides is

$$2(|\mathbf{u}|^2 + |\mathbf{v}|^2).$$

The diagonals are $\overrightarrow{PR} = \mathbf{u} + \mathbf{v}$ and $\overrightarrow{SQ} = \mathbf{u} - \mathbf{v}$, and so the lengths of the diagonals are $|\mathbf{u} + \mathbf{v}|$ and $|\mathbf{u} - \mathbf{v}|$. So the sum of the squares of the lengths of the diagonals is

$$|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$= |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 = 2(|\mathbf{u}|^2 + |\mathbf{v}|^2).$$
[25 marks]

Find an equation for the tangent line to the curve of intersection of the surfaces $z = x^2 - xy$ and $z = 1 - 2x + y^2$ at the point $(1, 1, 0)$.

Solution: Since we already have a point $(1, 1, 0)$ on the line, we only need a nonzero vector $\overrightarrow{w}$ in the direction of the line. Since the curve lies in both surfaces, $\overrightarrow{w}$ is orthogonal to the normals $\overrightarrow{u}$ and $\overrightarrow{v}$ of the tangent planes for each surface at $(1, 1, 0)$. Hence $\overrightarrow{w} = \overrightarrow{u} \times \overrightarrow{v}$ would do.

A normal for the surface $z = x^2 - xy$ at $(1, 1, 0)$ is given by

$$\overrightarrow{u} = (z_x, z_y, -1) = (2x - y, -x, -1) = (1, -1, -1)$$

A normal for the surface $z = 1 - 2x + y^2$ at $(1, 1, 0)$ is given by

$$\overrightarrow{v} = (z_x, z_y, -1) = (-2, 2y, -1) = (-2, 2, -1)$$

Thus, a vector in the direction of the line is

$$\overrightarrow{w} = \overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -2 & 2 & -1 \end{vmatrix} = (3, 3, 0)$$

So, a (parametric) equation for the line is $(1, 1, 0) + t(3, 3, 0)$, or $(1, 1, 0) + t(1, 1, 0)$.

Alternatively, one can find the equation of the line as the intersection of the two tangent planes.
3.

For each of the following functions \( f \) and points \( \mathbf{x}_0 \), determine whether or not \( f \) is continuous at \( \mathbf{x}_0 \). Justify your answers. You do NOT need to use \( \epsilon, \delta \) arguments.

(a) \( f : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x}_0 = (0, 0) \)

\[
f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}
\]

(b) \( f : \mathbb{R}^3 \to \mathbb{R}, \mathbf{x}_0 = (0, 0, 0) \)

\[
f(x, y, z) = \begin{cases} \frac{x^3yz^2}{x^6+y^6+z^6} & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0 & \text{if } (x, y, z) = (0, 0, 0) \end{cases}
\]

(c) \( f : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x}_0 = (0, 0) \)

\[
f(x, y) = \begin{cases} \frac{xy \sin(\frac{1}{x^2+y^2})}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
\]

Solution:

a. From class, we know that

\[
\lim_{(x,y)\to0} \frac{xy^2}{x^2+y^2} = 0
\]

But \( f(0) = 1 \). So, this function is not continuous at \( (0, 0) \).

(in fact, all we really need is that the limit as \( (x, y) \) approaches 0 along an axis is 0 and then since \( f(0) = 1 \neq 0 \), the function is not continuous at \( (0, 0) \)).
b. Along any axis, the function is identically zero and so \( f(x, y, z) \) approaches 0 as \((x, y, z) \to (0, 0, 0)\) along any axis. On the other hand, along the “diagonal” \( x = y = z \), the function is identically 1/3 and so \( f(x, y, z) \) approaches 1/3, as \((x, y, z) \to (0, 0, 0)\) along the diagonal. Hence the limit does not exist and so the function is not continuous at \((0, 0, 0)\).

(in fact, all we need is the limit along the diagonal, since 1/3 \( \neq f((0, 0, 0))\)).

c. 

\[
\left| \frac{xy \sin\left(\frac{1}{x^2+y^2}\right)}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{xy}{\sqrt{y^2}} \right| = |x||y|/|y| = |x|
\]

which approaches 0 as \((x, y)\) approaches 0. Thus,

\[
\lim_{(x,y) \to (0,0)} f(x, y) = 0 = f(0, 0)
\]

and so \( f \) is continuous at \((0, 0)\).
[25 marks]

(a) Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a differentiable function. Let
\[
w = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right).
\]
Show that
\[
x^2w_x + y^2w_y + z^2w_z = 0.
\]
at any point \((x_0, y_0, z_0)\) where \(x_0 \neq 0\), \(y_0 \neq 0\), and \(z_0 \neq 0\).

(b) Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and write \( f(x, y) = (g(x, y), k(x, y)) \). Assume that \( g \) and \( k \) are differentiable functions.

Let \( F(x, y) = g(f(x, y)) \).

Find \( F_x \) and \( F_y \) in terms of \( f(x, y) \) and partial derivatives of \( g(x, y) \) and \( k(x, y) \), explicitly indicating the points at which the partial derivatives are evaluated.

Solution:

a. Let \( u = \frac{y-x}{xy}, v = \frac{z-x}{xz} \). Then, \( w = f(u, v) \). Then, the intermediate variables are \( u, v \) and the primary variables are \( x, y, z \). Now,
\[
u_x = -1/x^2, u_y = 1/y^2, u_z = 0,
\]
and
\[
v_x = -1/x^2, v_y = 0, v_z = 1/z^2
\]
Then
\[
w_x = w_u u_x + w_v v_x = w_u(-1/x^2) + w_v(-1/x^2)
\]
\[
w_y = w_u u_y + w_v v_y = w_u(1/y^2)
\]
\[
w_z = w_u u_z + w_v v_z = w_v(1/z^2)
\]
So,
\[
x^2w_x + y^2w_y + z^2w_z = -(w_u + w_v) + w_u + w_v = 0.
\]
b. 

\[ F_x = g_x(f(x, y))g_x(x, y) + g_y(f(x, y))k_x(x, y) \]
\[ F_y = g_x(f(x, y))g_y(x, y) + g_y(f(x, y))k_y(x, y) \]