Lecture 29:

**Double Integrals**

Recall from 1-variable difference between definite integrals and indefinite integrals:

\[ \int f \, dx \text{ -vs- } \int_a^b f \, dx \]

function (anti-derivative) -vs- number

Idea: if \( f \geq 0 \), then \( \int_a^b f \, dx \) represents area under graph of \( f \).

Will define \( \int \int_D f(x, y) \, dA \) as a (definite) integral of a function \( f \), defined on a domain \( D \).

Will define the integral as a limit of Riemann sums in two dimensions:

Consider a function \( f(x, y) \) defined on a rectangle \( D = [a, b] \times [c, d] \).

1. Partition \([a, b]\) and \([c, d]\).
   
   \( a = x_0 < x_1 \ldots < x_{m-1} < x_m = b \)
   
   \( c = y_0 < y_1 \ldots < y_{n-1} < y_n = d \)

2. Yields partition \( P \) of \( D \) into \( mn \) sub-rectangles \( R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \).

3. Select arbitrary \((x_{ij}^*, y_{ij}^*) \in R_{ij}\).

4. Define Riemann sum:
\[ R(f, P) = \sum_{i,j} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij} \]

where \( \Delta A_{ij} = (\Delta x_{ij})(\Delta y_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1}) \).

Note: \( R(f, P) \) depends on partition and choice of points \((x_{ij}^*, y_{ij}^*)\).

5: Defn:

\( f \) is integrable on \( D \) if

\[ \lim_{||P|| \to 0} R(f, P) \]

exists and, if it does, define double integral, \( \int \int_D f \, dA \), to denote the limit.

Here, \( ||P|| = \max_{ij} \sqrt{(\Delta x_{ij})^2 + (\Delta y_{ij})^2} \), called mesh size or norm or maximum diameter.

Idea: if \( f \geq 0 \), then each term in a Riemann sum represents volume of a column over a small rectangle.

Add up volumes and take limit as mesh size goes to 0.

Result: \( \int \int_D f \, dA \) is the volume under the graph (between graph of \( f \) and \( xy=\text{plane} \)).

Q: Why do we want the mesh size to be defined in terms of the maximum diameter instead of the maximum area of the rectangles?

FORMAL definition of this limit is given on p. 817:
There exists a number $I$ s.t. given $\epsilon > 0$, there exists $\delta > 0$ such that if $||P|| < \delta$, then $|R(f, P) - I| < \epsilon$.

If this limit exists we denote $\int \int_D f(x, y)dA := I$ and say that $f$ is integrable over $D$.

Proposition: If $f$ is continuous, then it is integrable over any rectangle.

Rough idea: Given $\epsilon > 0$, choose $\delta > 0$ so small that if mesh size of partition $P$ is $< \delta$, then

$$\max f|_{R_{ij}} - \min f|_{R_{ij}} < \epsilon/(\text{Area}(D))$$

Then difference between biggest possible and least possible Riemann sum on $P$ (using different choices of $(x_{ij}^*, y_{ij}^*)$) is at most $\epsilon \sum_{ij} \Delta A_{ij}/(\text{Area}(D)) = \epsilon$.

Extension to other domains:

If $D$ is bounded and $f$ is bounded, then extend $f$ to $\hat{f}$ on $R$ by setting $\hat{f} = 0$ on $R \setminus D$ where $R$ is some rectangle containing $D$, and set

$$\int \int_D f dA := \int \int_R \hat{f} dA$$

provided the latter exists. It turns out that this is well-defined, independent of the encompassing rectangle.

Theorem 1: Let $f$ be a continuous function on a closed, bounded set $D$, with piecewise smooth boundary. Then $f$ is integrable over $D$.

List of properties of Double Integrals: p. 819, KNOW THEM!

Inspection: Examples 3, 4.

$$\int \int_D 4dA = 4(\text{area}(D))$$
Let $D$ be unit disk: $\{(x, y) : x^2 + y^2 \leq 1\}$.

By symmetry,

$$\int \int_D xdA = 0, \int \int_D \sin(x)dA = 0,$$

Let $D$ be unit square: $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

$$\int \int_D xdA = 1/2$$

(volume of a triangular prism)