Lecture 15:

Finish up proof of chain rule.

**GENERALIZE from real-valued functions of several variables to vector-valued functions of several variables.**

Consider a function \( \overline{f} : \mathbb{R}^n \to \mathbb{R}^m \) (sometimes, \( \overline{f} \) will be defined only on a sub-domain of \( \mathbb{R}^n \)):

\[
\overline{f} = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))
\]

\( \overline{f} \) may be regarded as \( m \)-real-valued functions of \( n \) real variables:

Examples:

1. \( m = 1, n = 1 \): 1-variable calculus; \( f(x) = \sin(x) \).

2. \( m = 1 \), general \( n \): one real-valued function of \( n \)-real variables; \( f(x) = \sin(x^2y) \).

3. \( n = 1 \), general \( m \): \( m \) real-values functions of a single variable:

\[
\overline{f}(x) = (\sin(x), e^x, \log(x))
\]

traces out a curve in \( \mathbb{R}^m \).

4. \( m = 3, n = 2 \):

\[
\overline{f}(x, y) = (xy, \sin(x^2y), e^{xy^3}).
\]

Defn: \( \overline{f} \) is **cts** at \( \overline{x}_0 \) if

\[
\lim_{\overline{x} \to \overline{x}_0} \overline{f}(\overline{x}) = \overline{f}(\overline{x}_0)
\]

Precise meaning of this limit: copy the \( \epsilon-\delta \) definition from the case \( m = 1 \):

\[
0 < |\overline{x} - \overline{x}_0| < \delta \Rightarrow |\overline{f}(\overline{x}) - \overline{f}(\overline{x}_0)| < \epsilon.
\]

Defn: \( \overline{f} \) is **diffble** at \( \overline{x}_0 \) if there is an \( m \times n \) matrix \( A \) s.t.

\[
\lim_{\overline{h} \to \overline{0}} \frac{|\overline{f}(\overline{x}_0 + \overline{h}) - \overline{f}(\overline{x}_0) - A\overline{h}|}{|\overline{h}|} = 0
\]
Defn: The Jacobian matrix $D\overline{f}(\overline{x}_0)$ of $\overline{f}$ at $\overline{x}_0$ is the matrix of first-order partial derivatives:

$$D\overline{f}(\overline{x}_0)_{ij} = \frac{\partial f_i}{\partial x_j}(\overline{x}_0), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$ 

$D\overline{f}(\overline{x}_0)$ is the Jacobian matrix evaluated at $\overline{x}_0$.

$D\overline{f}(\overline{x}_0)$ IS NOT a matrix-vector multiplication.

Theorem: If $\overline{f}$ is differentiable at $\overline{x}_0$, then the matrix $A = D\overline{f}(\overline{x}_0)$.

Examples of Jacobian matrices in Examples above:
1. $D\overline{f} = [\cos(x)]$
2. $D\overline{f} = [2xy \cos(x^2y), x^2 \cos(x^2y)]$
3. 

$$D\overline{f} = \begin{bmatrix}
\cos(x) \\
e^x \\
1/x
\end{bmatrix}$$

4. 

$$D\overline{f} = \begin{bmatrix}
y & x \\
2xy \cos(x^2y) & x^2 \cos(x^2y) \\
y^3e^{xy^3} & 3xy^2e^{xy^3}
\end{bmatrix}$$

Defn: $\overline{f}$ is continuously differentiable ($C^1$) at $\overline{x}_0 \in \mathbb{R}^n$ if each $\frac{\partial f_i}{\partial x_j}$ is continuous at $\overline{x}_0$, i.e., each element of the Jacobian matrix is continuous.

Theorem: 

$$C^1 \Rightarrow \text{differentiable} \Rightarrow \text{continuous} \quad \Downarrow
\quad \text{Jacobian matrix exists}$$

Theorem:

1. $\overline{f}$ is continuous if each $f_i$ is continuous, $i = 1, \ldots, m$
2. $\overline{f}$ is differentiable if each $f_i$ is differentiable, $i = 1, \ldots, m$
3. \( \overline{f} \) is \( C^1 \) iff each \( f_i \) is \( C^1 \), \( i = 1, \ldots, m \)

For part 1, note the difference between \( m = 1, n = 2 \) and \( m = 2, n = 1 \):

- \( f(x, y) = \frac{xy}{x^2+y^2} \) \( (x, y) \neq (0, 0) \)
  \( \frac{0}{0} \) \( (x, y) = (0, 0) \)

is not cts at \((0,0)\) even though both \( f(x,0) \) and \( f(0,y) \) are continuous at 0.

\( f(x, y) \) cannot be expressed by 2 functions of 1 variable.

- \( \overline{f}(x) = (\sin(x), e^x) \) is cts. exactly because both \( \sin(x) \) and \( e^x \) are cts.

\( \overline{f}(x) \) can be expressed by 2 functions of 1 variable.

Behaviour in range “decouples,” behaviour in domain does not “decouple.”