Math 226, HW7, Due on Friday, November 3

1. Section 12.8: 14, 17

2. Let \( \bar{A}(r, \theta) := (r \cos(\theta), r \sin(\theta)) \).
   
   (a) Show that \( \bar{A}(r, \theta) \) satisfies the hypotheses of the Inverse function theorem: for any point \((r_0, \theta_0) \) s.t. \( r_0 > 0 \), \( \bar{A} \) is \( C^1 \) in a nbhd. of \((r_0, \theta_0) \) and \( |D\bar{A}(r_0, \theta_0)| > 0 \).

   (b) Find an explicit inverse function \( \bar{B}(x, y) = (r(x, y), \theta(x, y)) \).

   (c) Compute \( \frac{\partial r}{\partial x} \) in two different ways and verify that you get the same answer:
      
      i. Using your explicit inverse function in part b.
      
      ii. Using the formula:

\[
\frac{\partial r}{\partial x} = -\frac{\partial (F,G)}{\partial (x,\theta)} \frac{\partial (F,G)}{\partial (r,\theta)}
\]

where

\[
F(x, y, r, \theta) = x - r \cos \theta, \quad G(x, y, r, \theta) = y - r \sin \theta
\]

Solution:

a. The partial derivatives of the component functions of \( \bar{A} \) are products of \( r \), \( \cos \theta \), and \( \sin \theta \), which are all \( C^1 \). Thus, \( \bar{A} \) is \( C^1 \).

The Jacobian

\[
|D\bar{A}| = \begin{vmatrix}
\cos(\theta) & -r \sin(\theta) \\
\sin(\theta) & r \cos(\theta)
\end{vmatrix} = r(\cos^2(\theta) + \sin^2(\theta)) = r > 0
\]

when \( r = r_0 > 0 \).

b. The idea is to solve for \( x \) and \( y \) in terms of \( r \) and \( \theta \) from the equations

\[
x = r \cos(\theta), \quad y = r \sin(\theta)
\]

\[
x^2 + y^2 = r^2 \quad \text{and so} \quad r = r(x, y) = \sqrt{x^2 + y^2}
\]

Let

\[
\theta = \theta(x, y) = \begin{cases}
\arccos(x/\sqrt{x^2 + y^2}) & y \geq 0 \\
-\arccos(x/\sqrt{x^2 + y^2}) & y < 0
\end{cases}
\]

Since cos is an even function (i.e., \( \cos(-\phi) = \cos(\phi) \)),

\[
\cos(\theta(x, y)) = \cos(\arccos(x/\sqrt{x^2 + y^2}) = x/\sqrt{x^2 + y^2}
\]

Thus,

\[
\sin(\theta(x, y)) = \pm\sqrt{1 - \cos^2(\theta(x, y))} = \pm\sqrt{1 - x^2/(x^2 + y^2)} = \pm y/\sqrt{x^2 + y^2}
\]
If $y \geq 0$, then $\theta(x, y) \in [0, \pi]$ and so $\sin(\theta(x, y)) \geq 0$ and so $\sin(\theta(x, y)) = y/\sqrt{x^2 + y^2}$.

If $y < 0$, then $\theta(x, y) \in [-\pi, 0]$ and so $\sin(\theta(x, y)) \leq 0$ and so $\sin(\theta(x, y)) = y/\sqrt{x^2 + y^2}$.

So, in either case, we have $\sin(\theta(x, y)) = y/\sqrt{x^2 + y^2}$.

Thus,

$$
\mathcal{A} \circ \mathcal{B}(x, y) = \mathcal{A}(\sqrt{x^2 + y^2}, \theta(x, y))
= (\sqrt{x^2 + y^2} \cos(\theta(x, y)), \sqrt{x^2 + y^2} \sin(\theta(x, y))) = (x, y).
$$

So, $\mathcal{B}$ is an explicit inverse of $\mathcal{A}$.

Note that, as defined, $\theta(x, y)$ is discontinuous at $(-1, 0)$. To obtain a $C^1$ inverse $\mathcal{B}$ in a nbhd. of $(x_0, y_0) := \mathcal{A}(r_0, \theta_0)$ s.t. $B(x_0, y_0) = (r_0, \theta_0)$ requires more work.

c i.

$$r = \sqrt{x^2 + y^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos(\theta)$$

Note: Here $r$ and $\theta$ are the dependent variables and $x$ and $y$ are the independent variables. So, we should expect that $\frac{\partial r}{\partial x}$ is a function of $x$ and $y$, not $\theta$. But $\theta$ is a function of $x$ and $y$, thus so is $\cos(\theta)$.

c ii.

$$\frac{\partial r}{\partial x} = -\frac{\frac{\partial (F, G)}{\partial (x, \theta)}}{\frac{\partial (F, G)}{\partial (r, \theta)}} = -\frac{(1)(-r \cos(\theta)) - (0)(r \sin(\theta))}{(-\cos(\theta))(-r \cos(\theta)) - (-\sin(\theta))(r \sin(\theta))}
= \frac{r \cos(\theta)}{r} = \cos(\theta).$$

3. For a subset $D \subseteq \mathbb{R}^n$, show that the following three properties are equivalent.

(a) $D$ is open
(b) $D \cap \partial D = \emptyset$
(c) $D = D \cap (\partial D)^c$

In your proof, you may use the fact, proven in class, that $D$ is closed iff $\partial D \subseteq D$.

$a \leftrightarrow b$: first observe that $\partial D = \partial (D^c)$ (because the definition of boundary of a set is exactly the same if one reverses the roles of $D$ and $D^c$). Thus,

$D$ is open iff $D^c$ is closed iff $\partial (D^c) \subseteq D^c$ iff $\partial D \subseteq D^c$ iff $D \cap (\partial D) = \emptyset$.

$b \leftrightarrow c$: this is a consequence of the fact that for any sets $A$ and $B$, $A \cap B = \emptyset$ iff $A \subseteq B^c$ iff $A = A \cap B^c$. 

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4. Which of the following subsets of $\mathbb{R}^2$ are open, closed, bounded? Give brief justifications.

(a) A straight line
(b) $\{(0,0), (1,0), (0,1)\}$
(c) The complement of the set in part b.
(d) $\{(x,y) : x > 0, y = \sin(1/x)\}$
(e) $A \cup B$ where $A$ is the closed unit disk and $B$ is the $x$-axis.
(f) $A \cap B$ where $A$ is the closed unit disk and $B$ is the $x$-axis.

Solution:

a. A straight line is closed but not open and not bounded. It is closed because it equals its own boundary. For the same reason it is not open. It is not open because it is not contained in any disk centered at the origin.

b. Any finite nonempty set $D$ is closed and bounded but not open. It is closed because it equals its own boundary and for the same reason is not open. It is bounded because it is contained in the disk of radius $r$ centered at the origin where $r$ is any number greater than the distance of any point in $D$ from the origin.

c. The complement of a finite nonempty set is open, bounded but not closed. It is open and not closed because its complement is the set in part b which is closed but not open. It is not bounded because its complement is bounded.

d. This set $D$ is not open, closed, or bounded. As shown in a previous homework, the boundary of $D$ is the union of itself and the set $\{(0,y) : -1 \leq y \leq 1\}$. Thus, $D$ does not contain all of its boundary and is thus not closed. It is not open since each point of $D$ is in its boundary. The set $D$ is not bounded because it is not contained in any disk centered at the origin.

e. and f. Both $A$ and $B$ are closed. The union and intersection of two closed sets is closed because the union and intersection of two open sets is open. To see the latter:

union of two open sets $C$ and $D$ is open: If $x \in C \cup D$, then WLOG we may assume that $x \in C$ and so for some $r > 0$, $B_r(x) \subseteq C \subseteq C \cup D$.

intersection of two open sets $C$ and $D$ is open: If $x \in C \cap D$, then for some $r, r' > 0$, $B_r(x) \subseteq C, B_{r'}(x) \subseteq D$; let $s = \min(r, r')$; then $B_s(x) \subseteq C \cap D$.

$A \cup B$ is not open because it contains one of its boundary points, e.g., $(1,0)$; $A \cap B$ is not open for the same reason.

$A \cup B$ is not bounded because it contains $B$, the $x$-axis which is not bounded.

$A \cap B$ is bounded because it is contained in $A$ which is bounded.

5. Section 13.1: 20, 22