Solution:

a. For each positive integer $n$, let $x_n = 1/(\pi n)$ and $y_n = 1/(\pi/2 + 2\pi n)$. Both sequences $x_n$ and $y_n$ approach zero as $n \to \infty$.

$$f(x_n) = \sin(\pi n) = 0 \quad \text{and} \quad f(y_n) = \sin(\pi/2 + 2\pi n) = 1$$

Thus, we get two different limits depending on how $x$ approaches $0$ (one answer when $x$ approaches along $x_n$ and another when $x$ approaches along $y_n$).

Thus, $\lim_{x \to 0} g_0(x)$ does not exist. In particular, $g_0$ is not continuous at $x_0 = 0$.

b. Since for $x \neq 0$, $|\sin(1/x)| \leq 1$, we have

$$|g_1(x)| \leq |x|$$

Thus $\lim_{x \to 0} g_1(x) = 0$ (perhaps you might want to quote the “sandwich” or “squeeze” theorem). Thus, $g_1$ is continuous at $x_0 = 0$

$$\lim_{h \to 0} \frac{g_1(h) - g_1(0)}{h} = \lim_{h \to 0} \sin(1/h) = \lim_{h \to 0} g_0(h)$$

which does not exist, as shown in part a. Thus, $g_1$ is not differentiable at $x_0 = 0$.

c. $\lim_{h \to 0} \frac{g_2(h) - g_2(0)}{h} = \lim_{h \to 0} h\sin(1/h) = \lim_{h \to 0} g_1(h) = 0$

as shown in part b. Thus, $g_2$ is differentiable at $x_0 = 0$, with $g_2'(0) = 0$.

For $x \neq 0$, we have, by the product rule

$$g_2'(x) = -\cos(1/x) + 2x\sin(1/x),$$

The limit as $x \to 0$ of the first term does not exist (similar to part a), and the limit as $x \to 0$ of the second term does exist (as in part b). Thus, $\lim_{x \to 0} g_2'(x)$ does not exist, so $g_2'$ is not continuously differentiable at $x_0 = 0$.

d. Similar to part c,

$$\lim_{h \to 0} \frac{g_3(h) - g_3(0)}{h} = \lim_{h \to 0} h^2\sin(1/h) = 0$$

and so $g_3'(0) = 0$. By the product rule,

$$g_3'(x) = -x\cos(1/x) + 3x^2\sin(1/x),$$

And both terms tend to $0$ as $x \to 0$ (similar to part b). Thus, $g_3'$ is continuously differentiable at $x_0 = 0$. 


Solution: \( \not\exists 4 \)

a. Let \( \bar{L} = \bar{f}(\bar{x}_0) \) and \( L_i = f_i(\bar{x}_0) \). By the given fact on limits, we get

\[
\lim_{\bar{x} \to \bar{x}_0} f(\bar{x}) = \bar{f}(\bar{x}_0) \iff \lim_{\bar{x} \to \bar{x}_0} f_i(\bar{x}) = f_i(\bar{x}_0)
\]

Thus, \( \bar{f} \) is continuous iff each \( f_i \) is continuous.

b. Note that the rows of the Jacobian matrix \( D\bar{f}(\bar{x}_0) \) are exactly the Jacobians \( Df_i(\bar{x}_0) \), and

\[
D\bar{f}(\bar{x}_0) = \begin{bmatrix}
(Df_1(\bar{x}_0))\bar{h}^T \\
\vdots \\
(Df_m(\bar{x}_0))\bar{h}^T
\end{bmatrix}
\]

(1)

Let \( \bar{L} = \bar{0} \) and \( L_i = 0 \). By the given fact on limits and (1), we get

\[
\lim_{\bar{h} \to \bar{0}} \frac{\bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0) - (D\bar{f}(\bar{x}_0))\bar{h}^T}{|\bar{h}|} = \bar{0}
\]

iff

\[
\text{for each } i, \lim_{\bar{h} \to \bar{0}} \frac{f_i(\bar{x}_0 + \bar{h}) - f_i(\bar{x}_0) - (Df_i(\bar{x}_0))\bar{h}^T}{|\bar{h}|} = 0
\]

Thus, \( \bar{f} \) is differentiable iff each \( f_i \) is differentiable.

c. Continuous differentiability of \( \bar{f} \) means continuity of each entry of \( D\bar{f}(\bar{x}_0) \). Continuous differentiability of \( f_i \) means continuity of each entry of \( Df_i(\bar{x}_0) \).

Since the \( i \)-th row of \( D\bar{f}(\bar{x}_0) \) is \( Df_i(\bar{x}_0) \), we obtain the result.