Math 226, HW5, due on Friday, October 20

1. Section 12.6: 1, 4, 17, 19

2. Chapter 12, Review Exercises: 4, 5

3. Let $a_1, \ldots, a_m$ be real numbers. Show that

\[ |a_i| \leq \sqrt{\sum_{j=1}^{m} a_j^2} \leq \sum_{i=1}^{m} |a_i| \]  \hspace{1cm} (1)

(the first inequality holds for all $i$).

Solution: For the first inequality, square both sides and observe:

\[ |a_i|^2 \leq \sum_{j=1}^{m} a_j^2 \]

For the second inequality, square both sides and observe:

\[ \sum_{j=1}^{m} a_j^2 \leq (\sum_{i=1}^{m} |a_i|)^2 \]

because the right side contains all terms in the left hand side and some cross terms as well.

4. Recall that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

\[ |\mathbf{x} - \mathbf{y}| := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \]

Write $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ as $\mathbf{f}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$. Let $\mathbf{L} = (L_1, \ldots, L_m) \in \mathbb{R}^m$.

Precise $\epsilon - \delta$ definition of $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L}$:

\[ \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } 0 < |\mathbf{x} - \mathbf{x}_0| < \delta \text{, then } |\mathbf{f}(\mathbf{x}) - \mathbf{L}| < \epsilon. \]

Show that

\[ \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{L} \text{ iff for each } i = 1, \ldots, m, \lim_{\mathbf{x} \to \mathbf{x}_0} f_i(\mathbf{x}) = L_i \]

Solution: By Problem 3 above,

\[ |f_i(\mathbf{x}) - L_i| \leq |\mathbf{f}(\mathbf{x}) - \mathbf{L}| \leq \sum_{i=1}^{m} |f_i(\mathbf{x}) - L_i| \]  \hspace{1cm} (2)
(the first equality holds for all $i$).

Assume:

$$\lim_{\overline{x} \to x_0} \overline{f}(\overline{x}) = \overline{L}.$$ 

Let $\epsilon > 0$. There exists $\delta > 0$ such that if $0 < |\overline{x} - \overline{x}_0| < \delta$, then $|\overline{f}(\overline{x}) - \overline{L}| < \epsilon$. By the first inequality in (2), for all $i$, $|f_i(\overline{x}) - L_i| < \epsilon$. Thus, for all $i$, $\lim_{\overline{x} \to \overline{x}_0} f_i(\overline{x}) = L_i$.

Conversely, assume for all $i$,

$$\lim_{\overline{x} \to \overline{x}_0} f_i(\overline{x}) = L_i$$

Let $\epsilon > 0$. For each $i$, there exists $\delta_i > 0$ such that if $0 < |\overline{x} - \overline{x}_0| < \delta_i$, then $|f_i(\overline{x}) - L_i| < \epsilon/m$. Let

$$\delta = \min\{\delta_1, \ldots, \delta_m\}$$

If $0 < |\overline{x} - \overline{x}_0| < \delta$, then each $|f_i(\overline{x}) - L_i| < \epsilon/m$ and thus $\sum_{i=1}^{m} |f_i(\overline{x}) - L_i| < \epsilon$. By the second inequality in (2), we obtain $|\overline{f}(\overline{x}) - \overline{L}| < \epsilon$. Thus, $\lim_{\overline{x} \to \overline{x}_0} \overline{f}(\overline{x}) = \overline{L}$. 

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