Math 226, HW4, Due on Friday, October 6, Solutions

1. Section 12.3: 7, 9, 11, 12, 22, 34
2. Section 12.4: 4, 5, 16

3. Sketch each of the following sets $D$ and identify its boundary $\partial D$. No proof is required.

(a) \[ \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\} \]

Solution:
\[ \partial D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \]

(b) \[ \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\} \]

Solution:
\[ \partial D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \]

(c) \[ \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 \leq 1\} \]

Solution:
\[ \partial D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 = 1\} \]

(d) \[ \{(x, y) \in \mathbb{R}^2 : x > 0, y = \sin(1/x)\} \]

Solution:
\[ \partial D = \{(x, y) \in \mathbb{R}^2 : x > 0, y = \sin(1/x)\} \cup \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\} \]

4. For each of the following, find the limit and prove, using the $\epsilon$-$\delta$ definition, that the limit exists.

(a) \[ \lim_{(x,y) \to (1,2)} x^2y^2 \]

Solution:
Let $\epsilon > 0$. We want to find $\delta > 0$ such that if $\sqrt{(x-1)^2 + (y-2)^2} < \delta$, then $|x^2y^2 - 4| < \epsilon$. We have:
\[ |x^2y^2 - 4| = |x^2y^2 - y^2 + y^2 - 4| \leq |x^2 - 1|y^2 + |y^2 - 4| = |x-1||x+1|y^2 + |y-2||y+2| \]

Since $\sqrt{(x-1)^2 + (y-2)^2} < \delta$, we have $|x - 1| < \delta$ and $|y - 2| < \delta$. Thus, the expression above is
\[ \leq \delta(|x + 1|y^2 + |y + 2|) \]

If $\delta < 1$, then $|x + 1| < 3$, $|y| \leq 3$ and $|y + 2| \leq 5$. Thus, the expression above is
\[ \leq 135\delta \]

So, choosing $\delta = \min(1, \epsilon/135)$, from the above computation we get
\[ |x^2y^2 - 4| < 135\delta < \epsilon. \]
(b) \( \lim_{(x,y) \to (0,0)} \frac{1}{x^2 + y^2 + 1} \)

Solution:
Let \( \epsilon > 0 \). We want to find \( \delta > 0 \) such that if \( \sqrt{x^2 + y^2} < \delta \), then \( |\frac{1}{x^2 + y^2 + 1} - 1| < \epsilon \).
We have:
\[
|\frac{1}{x^2 + y^2 + 1} - 1| = \frac{x^2 + y^2}{x^2 + y^2 + 1} \leq x^2 + y^2
\]
If \( \sqrt{x^2 + y^2} < \delta \), then
\[
x^2 + y^2 < \delta^2
\]
Choose \( \delta = \sqrt{\epsilon} \). Then
\[
|\frac{1}{x^2 + y^2 + 1} - 1| \leq x^2 + y^2 < \delta^2 = \epsilon.
\]
Note: another choice that works is \( \delta = \min(1, \epsilon) \).

(c) \( \lim_{(x,y) \to (0,0)} \frac{x^3y^2}{x^4 + y^4} \)

Solution:
Observe that \( (x^2 - y^2)^2 = x^4 + y^4 - 2x^2y^2 \). So,
\[
\frac{x^2y^2}{x^4 + y^4} = \frac{x^2y^2}{(x^2 - y^2)^2 + 2x^2y^2} \leq 1/2.
\]
So, \( |\frac{x^3y^2}{x^4 + y^4}| \leq (1/2)|x| \).

Let \( \delta = \epsilon \). If \( \sqrt{x^2 + y^2} < \delta \), then \( |x| < \delta = \epsilon \), and then
\[
|\frac{x^3y^2}{x^4 + y^4}| \leq (1/2)|x| < \epsilon/2 < \epsilon.
\]