34. The derivation of the equation of the hanging cable given in the text needs to be modified by replacing \( W = -\delta g \mathbf{j} \) with \( W = -\delta g \mathbf{x} \). Thus \( T_0 = \delta g x \), and the slope of the cable satisfies  
\[
\frac{dy}{dx} = \frac{\delta g x}{H} = ax
\]
where \( a = \delta g / H \). Thus  
\[
y = \frac{1}{2} ax^2 + C;
\]
the cable hangs in a parabola.

35. If \( y = \frac{1}{a} \cosh(ax) \), then \( y' = \sinh(ax) \), so  
\[
s = \int_0^x \sqrt{1 + \sinh^2(au)} \, du = \int_0^x \cosh(au) \, du
\]
\[
= \left. \frac{\sinh(au)}{a} \right|_0^x = \frac{1}{a} \sinh(ax).
\]

As shown in the text, the tension \( T \) at \( P \) has horizontal and vertical components that satisfy  
\[
T_h = H = \frac{\delta g}{a}\sinh(ax)
\]
and  
\[
T_v = \delta g x = \frac{\delta g}{a}\sinh(ax).
\]
Hence  
\[
|T| = \sqrt{T_h^2 + T_v^2} = \frac{\delta g}{a}\cosh(ax) = \delta g y.
\]

36. The cable hangs along the curve \( y = \frac{1}{a} \cosh(ax) \), and its length from the lowest point at \( x = 0 \) to the support tower at \( x = 45 \) m is 50 m. Thus  
\[
50 = \int_0^{45} \sqrt{1 + \sinh^2(ax)} \, dx = \frac{1}{a} \sinh(45a).
\]
The equation \( \sinh(45a) = 50a \) has approximate solution  
\[
a \approx 0.0178541.
\]
The vertical distance between the lowest point on the cable and the support point is  
\[
\frac{1}{a} (\cosh(45a) - 1) \approx 19.07 \text{ m}.
\]

37. The equation of the cable is of the form \( y = \frac{1}{a} \cosh(ax) \). At the point \( P \) where \( x = 10 \) m, the slope of the cable is  
\[
\sinh(10a) = \tan(55^\circ). 
\]
Thus  
\[
a = \frac{1}{10} \sinh^{-1}(\tan(55^\circ)) \approx 0.115423.
\]

The length of the cable between \( x = 0 \) and \( x = 10 \) m is  
\[
L = \int_0^{10} \sqrt{1 + \sinh^2(ax)} \, dx
\]
\[
= \frac{1}{a} \sinh(ax) \bigg|_0^{10} = \frac{1}{a} \sinh(10a) \approx 12.371 \text{ m}.
\]

**Section 10.3 The Cross Product in 3-Space (page 592)**

1. \((i - 2j + 3k) \times (3i + j - 4k) = 5i + 13j + 7k\)
2. \((j + 2k) \times (-i - j + k) = 3i - 2j + k\)
3. If \( A = (1, 2, 0) \), \( B = (1, 0, 2) \), and \( C = (0, 3, 1) \), then  
\[
\overrightarrow{AB} = -2j + 2k, \quad \overrightarrow{AC} = -i + j + k,
\]
and the area of triangle \( ABC \) is  
\[
\frac{\text{Area}}{2} = \frac{|-i - 2j - 2k|}{2} = \sqrt{6} \text{ sq. units}.
\]

4. A vector perpendicular to the plane containing the three given points is  
\[
(-ai + bj) \times (-ai + ck) = bci + acj + abk.
\]

A unit vector in this direction is  
\[
\frac{bci + acj + abk}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}.
\]
The triangle has area  
\[
\frac{1}{2} \sqrt{b^2c^2 + a^2c^2 + a^2b^2}.
\]

5. A vector perpendicular to \( i + j \) and \( j + 2k \) is  
\[
\pm (i + j) \times (j + 2k) = \pm(2i - 2j + k),
\]
which has length 3. A unit vector in that direction is  
\[
\pm \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right).
\]

6. A vector perpendicular to \( u = 2i - j - 2k \) and to \( v = 2i - 3j + k \) is the cross product  
\[
|u \times v| = \begin{vmatrix} i & j & k \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -7i - 6j - 4k,
\]
which has length  \( \sqrt{101} \). A unit vector with positive \( k \) component that is perpendicular to \( u \) and \( v \) is  
\[
\frac{-i}{\sqrt{101}} \times v = \frac{1}{\sqrt{101}} (7i + 6j + 4k).
\]
7. Since \( \mathbf{u} \) makes zero angle with itself, \( |\mathbf{u} \times \mathbf{u}| = 0 \) and \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \).

8. \( \mathbf{u} \times \mathbf{v} = \mathbf{i} j k \\
    \begin{vmatrix}
    u_1 & v_1 & \mathbf{i} \\
    u_2 & v_2 & \mathbf{j} \\
    u_3 & v_3 & \mathbf{k}
    \end{vmatrix}
    = - \mathbf{v} \times \mathbf{u}.
\)

9. \( (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{i} j k \\
    \begin{vmatrix}
    u_1 + v_1 & u_2 + v_2 & u_3 + v_3 \\
    w_1 & w_2 & w_3
    \end{vmatrix}
    = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.
\)

10. \( (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{i} j k \\
    \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
    \end{vmatrix}
    = \mathbf{i}(\mathbf{u} \times \mathbf{v}) - \mathbf{j}(\mathbf{u} \times \mathbf{v}) + \mathbf{k}(\mathbf{u} \times \mathbf{v}).
\)

11. \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{i} j k \\
    \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
    \end{vmatrix}
    = 0.
\)

12. Both \( \mathbf{u} = \cos \beta \mathbf{i} + \sin \beta \mathbf{j} \) and \( \mathbf{v} = \cos \alpha \mathbf{i} + \sin \alpha \mathbf{j} \) are unit vectors. They make angles \( \beta \) and \( \alpha \), respectively, with the positive x-axis, so the angle between them is \( |\alpha - \beta| \). They span a parallelogram (actually a rhombus) having area

\[ |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\alpha - \beta) = |\mathbf{u}||\mathbf{v}| \sin(\alpha - \beta). \]

13. Suppose that \( \mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0} \). Then

\[ \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{0}. \]

14. The base of the tetrahedron is a triangle spanned by \( \mathbf{v} \) and \( \mathbf{w} \), which has area

\[ A = \frac{1}{2} |\mathbf{v} \times \mathbf{w}|. \]

15. The tetrahedron with vertices \((1,0,0), (1,2,0), (2,2,0), \) and \((0,3,2)\) is spanned by \( \mathbf{u} = \mathbf{j}, \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \) and \( \mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}. \) By Exercise 14, its volume is

\[ V = \frac{1}{3} A h = \frac{1}{6} |\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) | = \frac{1}{6} \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
    \end{vmatrix} = \frac{4}{3} \text{ cu. units.} \]

16. Let the cube be as shown in the figure. The required parallelepiped is spanned by \( a\mathbf{i} + a\mathbf{j} + a\mathbf{k}, \) and \( a\mathbf{i} + a\mathbf{k}. \) Its volume is

\[ V = \begin{vmatrix}
    a & a & 0 \\
    0 & a & a \\
    a & 0 & a
    \end{vmatrix} = 2a^3 \text{ cu. units.} \]
17. The points \( A = (1, 1, -1) \), \( B = (0, 3, -2) \), \( C = (-2, 1, 0) \), and \( D = (k, 0, 2) \) are coplanar if \( (\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = 0 \).

Now

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2 & -1 \\
-3 & 2 & 1
\end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.
\]

Thus the four points are coplanar if

\[
2(k - 1) + 4(0 - 1) + 6(2 + 1) = 0,
\]

that is, if \( k = -6 \).

18. \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix}
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\
\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3
\end{vmatrix} = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \]

19. If \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq \mathbf{0} \), and \( \mathbf{x} = \lambda \mathbf{u} + \mu \mathbf{v} + \nu \mathbf{w} \), then

\[
\mathbf{x} \times (\mathbf{v} \times \mathbf{w}) = \lambda \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mu (\mathbf{v} \times (\mathbf{v} \times \mathbf{w})) + \nu (\mathbf{w} \times (\mathbf{v} \times \mathbf{w})) = \lambda \mathbf{u} \times (\mathbf{v} \times \mathbf{w}),
\]

Thus

\[
\lambda = \frac{\mathbf{x} \times (\mathbf{v} \times \mathbf{w})}{\mathbf{u} \times (\mathbf{v} \times \mathbf{w})}
\]

Since \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) \), we have, by symmetry,

\[
\mu = \frac{\mathbf{x} \times (\mathbf{w} \times \mathbf{u})}{\mathbf{u} \times (\mathbf{v} \times \mathbf{w})}, \quad \nu = \frac{\mathbf{x} \times (\mathbf{u} \times \mathbf{v})}{\mathbf{u} \times (\mathbf{v} \times \mathbf{w})}.
\]

20. If \( \mathbf{v} \times \mathbf{w} \neq \mathbf{0} \), then \( (\mathbf{v} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w}) \neq \mathbf{0} \). By the previous exercise, there exist constants \( \lambda \), \( \mu \) and \( \nu \) such that

\[
\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w} + \nu (\mathbf{v} \times \mathbf{w}),
\]

But \( \mathbf{v} \times \mathbf{w} \) is perpendicular to both \( \mathbf{v} \) and \( \mathbf{w} \), so

\[
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = 0 + 0 + \nu (\mathbf{v} \times \mathbf{w}) \times (\mathbf{v} \times \mathbf{w}).
\]

If \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = 0 \), then \( \nu = 0 \), and

\[
\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w}.
\]

21. \( \mathbf{u} = i + 2j + 3k \)

\( \mathbf{v} = 2i - 3j \)

\( \mathbf{w} = j - k \)

\( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times (3i + 2j + 2k) = -2i + 7j - 4k \)

\( (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (9i + 6j - 7k) \times \mathbf{w} = i + 9j + 9k \)

\( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \) lies in the plane of \( \mathbf{v} \) and \( \mathbf{w} \).

\( (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \) lies in the plane of \( \mathbf{u} \) and \( \mathbf{v} \).

22. \( \mathbf{u} \times \mathbf{v} \times \mathbf{w} \) makes sense in that it must mean \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \).

\( ((\mathbf{u} \times \mathbf{v}) \times \mathbf{w}) \) makes no sense since it is the cross product of a scalar and a vector.

\( \mathbf{u} \times \mathbf{v} \times \mathbf{w} \) makes no sense. It is ambiguous, since \( (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \) and \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \) are not in general equal.

23. As suggested in the hint, let the \( x \)-axis lie in the direction of \( \mathbf{v} \), and let the \( y \)-axis be such that \( \mathbf{w} \) lies in the \( xy \)-plane. Thus

\( \mathbf{v} = v_1 \mathbf{i} \)

\( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} \)

Thus \( \mathbf{v} \times \mathbf{w} = v_1 w_1 \mathbf{i} \times \mathbf{j} = v_1 w_1 \mathbf{k} \), and

\( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (u_1 v_1 + u_2 w_2) \mathbf{i} \times \mathbf{j} + (u_1 w_2 - u_2 w_1) \mathbf{j} \times \mathbf{k} \)

\( = u_1 v_1 w_1 \mathbf{k} + u_2 v_1 w_1 \mathbf{k} \)

\( = -u_1 v_1 w_1 \mathbf{k} - u_2 v_2 w_2 \mathbf{k} \)

But

\( (\mathbf{u} \times \mathbf{w}) \) \( - (\mathbf{u} \times \mathbf{v}) \)

\( = (u_1 w_1 + u_2 w_2) \mathbf{i} - u_1 v_1 w_1 \mathbf{k} - u_2 v_2 w_1 \mathbf{k} \)

\( = u_1 v_1 w_1 \mathbf{k} + u_2 v_1 w_1 \mathbf{k} \)

\( = -u_1 v_1 w_1 \mathbf{k} - u_2 v_2 w_2 \mathbf{k} \)

Thus \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{w}) \) \( - (\mathbf{u} \times \mathbf{v}) \).

24. If \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) are mutually perpendicular, then \( \mathbf{v} \times \mathbf{w} \) is parallel to \( \mathbf{u} \), so \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0} \). In this case,

\( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \pm |\mathbf{u}| |\mathbf{v}| |\mathbf{w}| \); the sign depends on whether \( \mathbf{u} \) and \( \mathbf{v} \times \mathbf{w} \) are in the same or opposite directions.

25. Applying the result of the previous exercise three times, we obtain

\[
\begin{align*}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) &= (\mathbf{u} \times \mathbf{w}) \mathbf{v} - (\mathbf{u} \times \mathbf{v}) \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \mathbf{u} - (\mathbf{v} \times \mathbf{u}) \mathbf{w} + (\mathbf{w} \times \mathbf{u}) \mathbf{v} - (\mathbf{w} \times \mathbf{v}) \mathbf{u} \\
&= 0.
\end{align*}
\]
26. If \( \mathbf{a} = -i + 2j + 3k \) and \( \mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \), then
\[
\mathbf{a} \times \mathbf{x} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 2 & 3 \\
x & y & z
\end{vmatrix}
= (2z - 3y)\mathbf{i} + (3x + z)y - (y + 2x)\mathbf{k}.
\]
provided \( 2z - 3y = 1 \), \( 3x + z = 5 \), and \( -y - 2x = -3 \).
This system is satisfied by \( x = t \), \( y = 3 - 2t \), \( z = 5 - 3t \),
for any real number \( t \). Thus
\[
x = t(3 - 2t)j + (5 - 3t)k
\]
gives a solution of \( \mathbf{a} \times \mathbf{x} = 1 + 5j - 3k \) for any \( t \).
These solutions constitute a line parallel to \( \mathbf{a} \).

27. Let \( \mathbf{a} = -i + 2j + 3k \) and \( \mathbf{b} = i + 5j \). If \( \mathbf{x} \) is a solution of
\( \mathbf{a} \times \mathbf{x} = \mathbf{b} \), then
\[
\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{x}) = 0.
\]
However, \( \mathbf{a} \cdot \mathbf{b} \neq 0 \), so there can be no such solution \( \mathbf{x} \).

28. The equation \( \mathbf{a} \times \mathbf{x} = \mathbf{b} \) can be solved for \( \mathbf{x} \) if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \).
The "only if" part is demonstrated in the previous solution. For the "if" part, observe that if
\( \mathbf{a} \cdot \mathbf{b} = 0 \) and \( \mathbf{x}_0 = (\mathbf{b} \times \mathbf{a})/|\mathbf{a}|^2 \), then by Exercise 23,
\[
\mathbf{a} \times \mathbf{x}_0 = \frac{1}{|\mathbf{a}|^2} \mathbf{a} \times (\mathbf{b} \times \mathbf{a}) = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b}}{|\mathbf{a}|^2} = \mathbf{b}.
\]
The solution \( \mathbf{x}_0 \) is not unique; as suggested by the example in Exercise 26, any multiple of \( \mathbf{a} \) can be added to it and the result will still be a solution. If \( \mathbf{x} = \mathbf{x}_0 + t\mathbf{a} \), then
\[
\mathbf{a} \times \mathbf{x} = \mathbf{a} \times \mathbf{x}_0 + t\mathbf{a} \times \mathbf{a} = \mathbf{b} + 0 = \mathbf{b}.
\]

Section 10.4 Planes and Lines

1. \( x^2 + y^2 + z^2 = z^2 \) represents a line in 3-space, namely
the \( z \)-axis.

b) \( x + y + z = x + y + z \) is satisfied by every point in
3-space.

c) \( x^2 + y^2 + z^2 = -1 \) is satisfied by no points in (real) 3-space.

2. The plane through \((0, 2, -3)\) normal to \( 4\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) has equation
\[
4(x - 0) - (y - 2) - 2(z + 3) = 0.
\]
or \( 4x - y - 2z = 4 \).

3. The plane through the origin having normal \( \mathbf{i} - \mathbf{j} + 2\mathbf{k} \) has
equation \( x - y + 2z = 0 \).

4. The plane passing through \((1, 2, 3)\), parallel to the plane
\( 3x + y - 2z = 15 \), has equation \( 3y + x - 2z = 3 + 2 - 6 \),
or \( 3x + y - 2z = -1 \).

5. The plane through \((1, 1, 0)\), \((2, 0, 2)\), and \((0, 3, 3)\) has normal
\[
(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = 7\mathbf{i} + 5\mathbf{j} - \mathbf{k}.
\]
It therefore has equation
\[
7(x - 1) + 5(y - 1) - (z - 0) = 0,
\]
or \( 7x + 5y - z = 12 \).

6. The plane passing through \((-2, 0, 0)\), \((0, 3, 0)\), and \((0, 0, 4)\)
has equation
\[
\frac{x}{-2} + \frac{y}{3} + \frac{z}{4} = 1,
\]
or \( 6x - 4y - 3z = -12 \).

7. The normal \( \mathbf{n} \) to a plane through \((1, 1, 1)\) and \((2, 0, 3)\)
must be perpendicular to the vector \( \mathbf{i} - \mathbf{j} + 2\mathbf{k} \) joining
these points. If the plane is perpendicular to the plane
\( x + 2y - 3z = 0 \), then \( \mathbf{n} \) must also be perpendicular to
\( \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \), the normal to this latter plane. Hence we can use
\[
\mathbf{n} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = -1 + 5\mathbf{j} + 3\mathbf{k}.
\]
The plane has equation
\[
-(x - 1) + 5(y - 1) + 3(z - 1) = 0,
\]
or \( x - 5y - 3z = -7 \).

8. Since \((-2, 0, -1)\) does not lie on \( x - 4y + 2z = -5 \), the
required plane will have an equation of the form
\[
2x + 3y - z + \lambda(x - 4y + 2z + 5) = 0
\]
for some \( \lambda \). Thus
\[
-4 + 1 + \lambda(-2 - 2 + 5) = 0,
\]
so \( \lambda = 3 \). The required plane is \( 5x - 9y + 5z = -15 \).

9. A plane through the line \( x + y = 2 \), \( y - z = 3 \) has equation
of the form
\[
x + y - 2 + \lambda(y - z - 3) = 0.
\]
This plane will be perpendicular to \( 2x + 3y + 4z = 5 \) if
\[
(2)(1) + (1 + \lambda)(3) - (\lambda)(4) = 0,
\]
that is, if \( \lambda = 5 \). The equation of the required plane is
\[
x + 6y - 5z = 17.
\]
10. Three distinct points will not determine a unique plane through them if they all lie on a straight line. If the points have position vectors \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{r}_3 \), then they will all lie on a straight line if

\[
(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) = \mathbf{0}.
\]

11. If the four points have position vectors \( \mathbf{r}_i \) (1 \( \leq i \leq 4 \)), then they are coplanar if, for example,

\[
(\mathbf{r}_2 - \mathbf{r}_1) \times \left[(\mathbf{r}_3 - \mathbf{r}_1) \times (\mathbf{r}_4 - \mathbf{r}_1)\right] = \mathbf{0}
\]

(or if they satisfy any similar such condition that asserts that the tetrahedron whose vertices they are has zero volume).

12. \( x + y + z = \lambda \) is the family of all (parallel) planes normal to the vector \( \mathbf{i} + \mathbf{j} + \mathbf{k} \).

13. \( x + \lambda y + \lambda z = \lambda \) is the family of all planes containing the line of intersection of the planes \( x = 0 \) and \( y + z = 1 \), except the plane \( y + z = 1 \) itself. All these planes pass through the points (0, 1, 0) and (0, 0, 1).

14. The distance from the planes

\[ \lambda x + \sqrt{1 - \lambda^2} y = 1 \]

to the origin is \( 1/\sqrt{\lambda^2 + 1 - \lambda^2} = 1 \). Hence the equation represents the family of all vertical planes at distance 1 from the origin. All such planes are tangent to the cylinder \( x^2 + y^2 = 1 \).

15. The line through (1, 2, 3) parallel to \( 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k} \) has equations given in vector parametric form by

\[
\mathbf{r} = (1 + 2t)\mathbf{i} + (2 - 3t)\mathbf{j} + (3 - 4t)\mathbf{k},
\]

or in scalar parametric form by

\[
x = 1 + 2t, \quad y = 2 - 3t, \quad z = 3 - 4t,
\]

or in standard form by

\[
\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{4}.
\]

16. The line through (1, 0, 1) perpendicular to the plane \( 2x - y + 7z = 12 \) is parallel to the normal vector \( 2\mathbf{i} - \mathbf{j} + 7\mathbf{k} \) to that plane. The equations of the line are, in vector parametric form,

\[
\mathbf{r} = (-1 + 2t)\mathbf{i} - \mathbf{j} + (1 + 7t)\mathbf{k},
\]

or in scalar parametric form,

\[
x = -1 + 2t, \quad y = -1, \quad z = 1 + 7t.
\]

17. A line parallel to the line with equations

\[
x + 2y - z = 2, \quad 2x - y + 4z = 5
\]
is parallel to the vector

\[
(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = (1 + 2\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.
\]

Since the line passes through the origin, it has equations

\[
x = 7t, \quad y = -6t, \quad z = -5t \quad \text{(vector parametric)}
\]

\[
x = \frac{y}{-6} = \frac{z}{-5} \quad \text{(scalar parametric)}
\]

18. A line parallel to \( x + y = 0 \) and \( x - y + 2z = 0 \) is parallel to the cross product of the normal vectors to these two planes, that is, to the vector

\[
(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 2(\mathbf{i} - \mathbf{j} - \mathbf{k}).
\]

Since the line passes through \( (2, -1, -1) \), its equations are, in vector parametric form

\[
\mathbf{r} = (2 + t)\mathbf{i} - (1 + t)\mathbf{j} - (1 + t)\mathbf{k},
\]

or in scalar parametric form

\[
x = 2 + t, \quad y = -(1 + t), \quad z = -(1 + t),
\]

or in standard form

\[
x - 2 = -(y + 1) = -(z + 1).
\]

19. A line making equal angles with the positive directions of the coordinate axes is parallel to the vector \( \mathbf{i} + \mathbf{j} + \mathbf{k} \).

If the line passes through the point (1, 2, -1), then it has equations

\[
\mathbf{r} = (1 + t)\mathbf{i} + (2 + t)\mathbf{j} + (-1 + t)\mathbf{k} \quad \text{(vector parametric)}
\]

\[
x = 1 + t, \quad y = 2 + t, \quad z = -1 + t \quad \text{(scalar parametric)}
\]

\[
x - 1 = y - 2 = z + 1 \quad \text{(standard form)}.
\]

20. The line \( \mathbf{r} = (1 - 2t)\mathbf{i} + (4 + 3t)\mathbf{j} + (9 - 4t)\mathbf{k} \) has standard form

\[
\frac{x - 1}{-2} = \frac{y - 4}{3} = \frac{z - 9}{-4}.
\]
21. The line \( \begin{align*}
  x &= 4 - 5t \\
  y &= 3t \\
  z &= 7
\end{align*} \) has standard form

\[
\frac{x - 4}{-5} = \frac{y}{3} = \frac{z - 7}{1}.
\]

22. The line \( \begin{align*}
  x - 2y + 3z &= 0 \\
  2x + 3y - 4z &= 4
\end{align*} \) is parallel to the vector

\[
(i - 2j + 3k) \times (2i + 3j - 4k) = -4i + 10j + 7k.
\]

We need a point on this line. Putting \( z = 0 \), we get

\[
\begin{align*}
  x - 2y &= 0, \\
  2x + 3y &= 4.
\end{align*}
\]

The solution of this system is \( y = 4/7, x = 8/7 \). A possible standard form for the given line is

\[
\begin{align*}
  \frac{x - 8}{10} &= \frac{y - 4}{14} = \frac{z}{7}.
\end{align*}
\]

though, of course, this answer is not unique as the coordinates of any point on the line could have been used.

23. The equations

\[
\begin{align*}
  x &= x_1 + t(x_2 - x_1) \\
  y &= y_1 + t(y_2 - y_1) \\
  z &= z_1 + t(z_2 - z_1)
\end{align*}
\]

certainly represent a straight line. Since

\[
(x, y, z) = (x_1, y_1, z_1) \text{ if } t = 0, \quad \text{and} \quad (x, y, z) = (x_2, y_2, z_2) \text{ if } t = 1,
\]

the line must pass through \( P_1 \) and \( P_2 \).

24. The point on the line corresponding to \( t = -1 \) is the point

\( P_0 \) such that \( P_0 \) is midway between \( P_1 \) and \( P_2 \).

The point on the line corresponding to \( t = 1/2 \) is the midpoint between \( P_1 \) and \( P_2 \).

The point on the line corresponding to \( t = 2 \) is the point

\( P_4 \) such that \( P_4 \) is the midpoint between \( P_1 \) and \( P_4 \).

25. Let \( r_i \) be the position vector of \( P_i \) (\( 1 \leq i \leq 4 \)). The line \( P_1 P_2 \) intersects the line \( P_3 P_4 \) in a unique point if the four points are coplanar, and \( P_1 P_2 \) is not parallel to \( P_3 P_4 \). It is therefore sufficient that

\[
(r_2 - r_1) \times (r_4 - r_3) \neq 0, \quad \text{and} \quad (r_3 - r_1) \cdot [(r_2 - r_1) \times (r_4 - r_3)] = 0.
\]

(Other similar answers are possible.)

26. The distance from \((0,0,0)\) to \( x + 2y + 3z = 4 \) is

\[
\frac{4}{\sqrt{14}} = \frac{4}{\sqrt{14}} \text{ units.}
\]

27. The distance from \((1,2,0)\) to \( 3x - 4y - 5z = 2 \) is

\[
\frac{|3 \cdot 1 - 8 \cdot 0 - 2|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{7}{5\sqrt{2}} \text{ units.}
\]

28. A vector parallel to the line \( x + y + z = 0, 2x - y - 5z = 1 \) is

\[
a = (i + j + k) \times (2i - j - 5k) = -4i + 7j - 3k.
\]

We need a point on this line: if \( z = 0 \) then \( x + y = 0 \) and \( 2x - y = 1 \), so \( x = 1/3 \) and \( y = -1/3 \). The position vector of this point is

\[
r_1 = \frac{1}{3}i - \frac{1}{3}j.
\]

The distance from the origin to the line is

\[
s = \frac{|r_1 \cdot a|}{|a|} = \frac{|i + j + k|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{3}{\sqrt{3}} = \frac{\sqrt{3}}{3} \text{ units.}
\]

29. The line \( \begin{align*}
  x + 2y &= 3 \\
  y + 2z &= 3
\end{align*} \) contains the points \((1,1,1)\) and \((3,0,2/3)\), so is parallel to the vector \( 2i - j + \frac{1}{2}k \), or to \( 4i - 2j + k \).

The line \( x + y + z = 6 \) contains the points \((-5,11,0)\) and \((-1,5,2)\), and so is parallel to the vector \( 4i - 6j + 2k \), or to \( 2i - 3j + k \).

Using the values

\[
\begin{align*}
  r_1 &= i + j + k, \\
  a_1 &= 4i - 2j + k, \\
  r_2 &= -i + 5j + 2k, \\
  a_2 &= 2i - 3j + k,
\end{align*}
\]

we calculate the distance between the two lines by the formula in Section 10.4 as

\[
s = \frac{|(r_2 - r_1) \cdot (a_1 \times a_2)|}{|a_1 \times a_2|} = \frac{|(2i - 4j - k) \cdot (i - 2j - 8k)|}{|i - 2j - 8k|} = \frac{18}{\sqrt{69}} \text{ units.}
\]

30. The line \( x - 2 = \frac{y + 3}{2} = \frac{z - 1}{4} \) passes through the point \((2,3,1)\), and is parallel to \((1 + 2j + 4k)\).

The plane \( 2y - z = 1 \) has normal \( n = 2j - k \).

Since \( a \cdot n = 0 \), the line is parallel to the plane.

The distance from the line to the plane is equal to the distance from \((2,3,1)\) to the plane \( 2y - z = 1 \), so is

\[
D = \frac{|-6 - 1 - 1|}{\sqrt{4 + 1}} = \frac{8}{\sqrt{5}} \text{ units.}
\]
31. \((1 - \lambda)(x - x_0) = \lambda(y - y_0)\) represents any line in the \(xy\)-plane passing through \((x_0, y_0)\). Therefore, in 3-space the pair of equations
\[
(1 - \lambda)(x - x_0) = \lambda(y - y_0), \quad z = z_0
\]
represents all straight lines in the plane \(z = z_0\) which pass through the point \((x_0, y_0, z_0)\).

32. \(\frac{x - x_0}{\sqrt{1 - \lambda^2}} = \frac{y - y_0}{\lambda} = z - z_0\) represents all lines through
\((x_0, y_0, z_0)\) parallel to the vectors
\[
a = \sqrt{1 - \lambda^2}\hat{i} + \lambda\hat{j} + \hat{k}.
\]
All such lines are generators of the circular cone
\[
(z - z_0)^2 = (x - x_0)^2 + (y - y_0)^2,
\]
so the given equations specify all straight lines lying on that cone.

33. The equation
\[
(A_1x + B_1y + C_1z + D_1)(A_2x + B_2y + C_2z + D_2) = 0
\]
is satisfied if either \(A_1x + B_1y + C_1z + D_1 = 0\) or \(A_2x + B_2y + C_2z + D_2 = 0\), that is, if \((x, y, z)\) lies on either of these planes. It is not necessary that the point lie on both planes, so the given equation represents all the points on each of the planes, not just those on the line of intersection of the planes.

Section 10.5 Quadric Surfaces (page 603)

1. \(x^2 + 4y^2 + 9z^2 = 36\)
\[
\frac{x^2}{6^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1
\]
This is an ellipsoid with centre at the origin and semi-axes 6, 3, and 2.

2. \(x^2 + y^2 + 4z^2 = 4\) represents an oblate spheroid, that is, an ellipsoid with its two longer semi-axes equal. In this case the longer semi-axes have length 2, and the shorter one (in the \(z\) direction) has length 1. Cross-sections in planes perpendicular to the \(z\)-axis between \(z = -1\) and \(z = 1\) are circles.

3. \(2x^2 + 2y^3 + 2z^2 - 4x + 8y - 12z + 27 = 0\)
\[
2(x^2 - 2x + 1) + 2(y^2 + 4y + 4) + 2(z^2 - 6z + 9) = -27 + 2 + 8 + 18
\]
\[
(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = \frac{1}{2}
\]
This is a sphere with radius \(1/\sqrt{2}\) and centre \((1, -2, 3)\).

4. \(x^2 + 4y^2 + 9z^2 + 4x - 8y = 8\)
\[
(x + 2)^2 + 4(y - 1)^2 + 9z^2 = 8 + 8 = 16
\]
\[
\frac{(x + 2)^2}{4} + \frac{(y - 1)^2}{1} + \frac{z^2}{(4/3)^2} = 1
\]
This is an ellipsoid with centre \((-2, 1, 0)\) and semi-axes 4, 2, and 4/3.

5. \(z = x^2 + 2y^2\) represents an elliptic paraboloid with vertex at the origin and axis along the positive \(z\)-axis. Cross-sections in planes \(z = k > 0\) are ellipses with semi-axes \(\sqrt{k}\) and \(\sqrt{2/k}\).

6. \(z = x^2 - 2y^2\) represents a hyperbolic paraboloid.

7. \(x^2 - y^2 - z^2 = 4\) represents a hyperboloid of two sheets with vertices at \((\pm2, 0, 0)\) and circular cross-sections in planes \(x = k\), where \(|k| > 2\).