Definitions: Continuity, Jacobian, Differentiability, Continuous Differentiability

Let \( \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m \) and \( \mathbf{x}_0 \in \mathbb{R}^n \).

We can write \( \mathbf{f} = (f_1, \ldots, f_m) \) where each \( f_i : \mathbb{R}^n \to \mathbb{R} \). Note that the \( f_i \) are the component functions of \( \mathbf{f} \) and not partial derivatives.

For example,

\[
\mathbf{f}(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3)) = (e^{x_1 x_2 x_3}, x_1^2 x_2^3 x_3^4)
\]

is a function from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \).

Alternatively, we can write:

\[
\mathbf{f} = \begin{bmatrix}
  f_1 \\
  \vdots \\
  f_m
\end{bmatrix}
\]

In this notation, the example above is written

\[
\mathbf{f}(x_1, x_2, x_3) = \begin{bmatrix}
  f_1(x_1, x_2, x_3) \\
  f_2(x_1, x_2, x_3)
\end{bmatrix} = \begin{bmatrix}
  e^{x_1 x_2 x_3} \\
  x_1^2 x_2^3 x_3^4
\end{bmatrix}
\]

Defn: \( \mathbf{f} \) is continuous at \( \mathbf{x}_0 \) if:

\[
\lim_{\mathbf{h} \to 0} \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) = \mathbf{f}(\mathbf{x}_0)
\]

\( m = 1, n = 1 \):

\[
\lim_{\mathbf{h} \to 0} f(x_0 + \mathbf{h}) = f(x_0)
\]

\( m = 1, n = 2 \):

\[
\lim_{(h,k) \to (0,0)} f(x_0 + h, y_0 + k) = f(x_0, y_0)
\]

\( m = 2, n = 1 \):

\[
\lim_{h \to 0} \begin{bmatrix}
  f_1(x_0 + h) \\
  f_2(x_0 + h)
\end{bmatrix} = \begin{bmatrix}
  f_1(x_0) \\
  f_2(x_0)
\end{bmatrix}
\]

Defn: Jacobian matrix: is the \( m \times n \) matrix: \( D\mathbf{f}(\mathbf{x}_0) \) defined by:

\[
(D\mathbf{f}(\mathbf{x}_0))_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)
\]

\( m = 1, n = 1 \): \( Df(x_0) = f'(x_0) \)

\( m = 1, n = 2 \): \( Df(x_0, y_0) = \left[ \frac{\partial f}{\partial x}(x_0, y_0) \frac{\partial f}{\partial y}(x_0, y_0) \right] \)

\( m = 2, n = 1 \): \( Df(x_0) = \begin{bmatrix}
  f_1'(x_0) \\
  f_2'(x_0)
\end{bmatrix} \)
Defn: $\vec{f}$ is differentiable at $\vec{x}_0$ if:

$$\lim_{\vec{h} \to \vec{0}} \frac{\vec{f}(\vec{x}_0 + \vec{h}) - \vec{f}(\vec{x}_0) - (D\vec{f}(\vec{x}_0)) \vec{h}}{|\vec{h}|} = \vec{0}.$$

$m = 1, n = 1$:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{h} = 0.$$

$m = 1, n = 2$:

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \left[ \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right] \begin{bmatrix} h \\ k \end{bmatrix}}{\sqrt{h^2 + k^2}} = 0.$$

$m = 2, n = 1$:

$$\lim_{h \to 0} \frac{\left[ f_1(x_0 + h) \right]}{h} - \frac{\left[ f_1(x_0) \right]}{h} - \frac{\left[ f_1'(x_0) \right]}{h} = 0.$$

Defn: $\vec{f}$ is continuously differentiable at $\vec{x}_0$ if each partial derivative $\frac{\partial f}{\partial x_j}$ is continuous at $\vec{x}_0$.

$m = 1, n = 1$: $f'(x)$ is continuous at $x = x_0$.

$m = 1, n = 2$: $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(x_0, y_0)$

$m = 2, n = 1$: $f_1'(x)$ and $f_2'(x)$ are continuous at $x = x_0$.

Defn: To say that $\vec{f}$ is (continuous, differentiable, continuously differentiable) in a neighbourhood of $\vec{x}_0$ means that for some $r > 0$ and all $\vec{x} \in B_r(\vec{x}_0)$, $\vec{f}$ is (continuous, differentiable, continuously differentiable) at $\vec{x}$. 

2