Thus \( f(x, y) \) has different constant values along different rays from the origin unless \( a = c = 0 \) and \( b \neq 0 \). If this condition is not satisfied, \( \lim_{(x, y) \to (0, 0)} f(x, y) \) does not exist. If the condition is satisfied, then \( \lim_{(x, y) \to (0, 0)} f(x, y) = 1/b \) does exist.

20. \( f(x, y) = \frac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)} \) cannot be defined at \((0, 0)\) so as to become continuous there, because \( f(x, y) \) has no limit as \((x, y) \to (0, 0)\). To see this, observe that \( f(x, 0) = 0 \), so the limit must be 0 if it exists at all. However,

\[
f(x, x) = \frac{\sin^4 x}{2\sin^2(x^2)}
\]

which approaches \(1/2\) as \(x \to 0\) by l'Hôpital's Rule or by using Maclaurin series.

21. The graphing software is unable to deal effectively with the discontinuity at \((x, y) = (0, 0)\) so it leaves some gaps and rough edges near the \(z\)-axis. The surface lies between a ridge of height 1 along \(y = x\) and a ridge of height \(-1\) along \(y = -x\). It appears to be creased along the \(z\)-axis. The level curves are straight lines through the origin.

22. The graphing software is unable to deal effectively with the discontinuity at \((x, y) = (0, 0)\) so it leaves some gaps and rough edges near the \(z\)-axis. The surface lies between a ridge along \(y = x^2\), \(z = 1\), and a ridge along \(y = -x^2\), \(z = -1\). It appears to be creased along the \(z\)-axis. The level curves are parabolas \(y = kx^2\) through the origin. One of the families of rulings on the surface is the family of contours corresponding to level curves.

23. The graph of a function \( f(x, y) \) that is continuous on region \(R\) in the \(xy\)-plane is a surface with no breaks or tears in it and that intersects each line parallel to the \(z\)-axis through a point \((x, y)\) of \(R\) at exactly one point.

24. (a) We say that \( \lim_{x \to a} f(x) = L \) provided that (i) every open interval containing \(a\) contains at least one point of the domain of \(f\) different from \(a\), and (ii) if for every \(\epsilon > 0\) there exists \(\delta > 0\) depending on \(\epsilon\) such that if \(x\) is in the domain of \(f\) and satisfies \(0 < |x - a| < \delta\), then \(|f(x) - L| < \epsilon\).

(b) There are no points in the domain of \(f\) to the right of 1 or between 1/2 and 1 so condition (i) of the definition is not satisfied and \( \lim_{x \to 1} f(x) \) does not exist. If \((a, b)\) is any open interval containing 0, then \(b > 0\). If integer \(n > 1/b\), then \(1/n < b\) and so \((a, b)\) contains a point of the domain of \(f\). If \(\epsilon > 0\), let \(\delta = \epsilon\). If \(1/n < \delta\), then

\[
|f\left(\frac{1}{n}\right) - 1| = \left|\frac{n-1}{n} - 1\right| = \frac{1}{n} < \epsilon.
\]

Thus \( \lim_{x \to a} f(x) \) exists and equals 1.

(c) Since every open interval on the real line contains irrational numbers that are not in the domain of \(f\), the conditions of Definition 8 in Section 1.5 are not met, so neither of the limits above can exist under that definition.

Section 12.3 Partial Derivatives (page 696)

1. \( f(x, y) = x - y + 2 \)
   
   \( f_1(x, y) = 1 = f_1(3, 2), \quad f_2(x, y) = -1 = f_2(3, 2) \).
2. \( f(x, y) = xy + x^2 \),
\[ f_1(x, y) = y + 2x, \quad f_2(x, y) = x, \]
\[ f_1(2, 0) = 0, \quad f_2(2, 0) = 2. \]

3. \( f(x, y, z) = x^3y^2z^5 \),
\[ f_1(x, y, z) = 3x^2y^2z^5, \quad f_2(x, y, z) = 4x^3y^2z^4, \quad f_3(x, y, z) = 5x^3y^4z^4, \]
\[ f_1(0, -1, -1) = 0, \quad f_2(0, -1, -1) = 0, \quad f_3(0, -1, -1) = 0. \]

4. \( g(x, y, z) = \frac{xz}{y + z} \),
\[ g_1(x, y, z) = \frac{z}{y + z}, \quad g_2(x, y, z) = -\frac{xz}{(y + z)^2}, \quad g_3(x, y, z) = \frac{xy}{(y + z)^2}, \]
\[ g_1(1, 1, 1) = \frac{1}{2}, \quad g_2(1, 1, 1) = -\frac{1}{4}, \quad g_3(1, 1, 1) = \frac{1}{4}. \]

5. \( z = \tan^{-1}\left( \frac{y}{x} \right) \)
\[ \frac{\partial z}{\partial x} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \left( -\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2}, \]
\[ \frac{\partial z}{\partial y} = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \left( \frac{1}{x} \right) = \frac{x}{x^2 + y^2}, \]
\[ \frac{\partial z}{\partial x} \bigg|_{(1, -1)} = -\frac{1}{2}, \quad \frac{\partial z}{\partial y} \bigg|_{(1, -1)} = \frac{1}{2}. \]

6. \( w = \ln(1 + e^{xyz}) \),
\[ \frac{\partial w}{\partial x} = \frac{yze^{xyz}}{1 + e^{xyz}}, \quad \frac{\partial w}{\partial y} = \frac{xze^{xyz}}{1 + e^{xyz}}, \quad \frac{\partial w}{\partial z} = \frac{xe^{xyz}}{1 + e^{xyz}}. \]
At \( (2, 0, -1) \): \( \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = -1, \quad \frac{\partial w}{\partial z} = 0. \)

7. \( f(x, y) = \sin(x\sqrt{y}) \),
\[ f_1(x, y) = \sqrt{y} \cos(x\sqrt{y}), \quad f_1 \left( \frac{\pi}{3}, 4 \right) = -1, \]
\[ f_2(x, y) = \frac{x}{2\sqrt{y}} \cos(x\sqrt{y}), \quad f_2 \left( \frac{\pi}{3}, 4 \right) = -\frac{\pi}{24}. \]

8. \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \),
\[ f_1(x, y) = -\frac{1}{2}(x^2 + y^2)^{-3/2}(2x) = -\frac{x}{(x^2 + y^2)^{3/2}}, \]
By symmetry, \( f_2(x, y) = -\frac{y}{(x^2 + y^2)^{3/2}} \),
\[ f_1(-3, 4) = \frac{3}{125}, \quad f_2(-3, 4) = -\frac{4}{125}. \]

9. \( w = x^y \ln z \),
\[ \frac{\partial w}{\partial x} = y \ln z x^{y-1}, \quad \frac{\partial w}{\partial y} \bigg|_{(e, 2, e)} = 2e, \]
\[ \frac{\partial w}{\partial y} = \ln x \ln z x^{y-1}, \quad \frac{\partial w}{\partial z} \bigg|_{(e, 2, e)} = e^2, \]
\[ \frac{\partial w}{\partial z} = \frac{y}{z} \ln x x^{y-1}, \quad \frac{\partial w}{\partial z} \bigg|_{(e, 2, e)} = 2e. \]

10. If \( g(x_1, x_2, x_3, x_4) = \frac{x_1 - x_2^2}{x_3 + x_4} \), then
\[ g_1(x_1, x_2, x_3, x_4) = \frac{1}{x_3 + x_4}, \quad g_1(3, 1, -1, 2) = \frac{1}{3}, \]
\[ g_2(x_1, x_2, x_3, x_4) = \frac{-2x_2}{x_3 + x_4}, \quad g_2(3, 1, -1, 2) = \frac{2}{3}, \]
\[ g_3(x_1, x_2, x_3, x_4) = \frac{x_2 - x_1}{(x_3 + x_4)^2}, \quad g_3(3, 1, -1, 2) = \frac{2}{9}, \]
\[ g_4(x_1, x_2, x_3, x_4) = \frac{(x_2 - x_1)^2 x_4}{(x_3 + x_4)^2}, \quad g_4(3, 1, -1, 2) = \frac{8}{9}. \]

11. \( f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\
0, & \text{if } (x, y) = (0, 0) \end{cases} \)
\[ f_1(0, 0) = \lim_{h \to 0} \frac{2h^3 - 0}{h(h^2 + 0)} = 2, \]
\[ f_2(0, 0) = \lim_{k \to 0} \frac{-3^3 - 0}{k(0 + 3k^2)} = -\frac{1}{3}. \]

12. \( f(x, y) = \begin{cases} \frac{x^2 - 2y^2}{x - y}, & \text{if } x \neq y \\
0, & \text{if } x = y \end{cases} \)
\[ f_1(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1, \]
\[ f_2(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{2k}{k} = 2. \]

13. \( f(x, y) = x^2 - y^2 \), \( f(-2, 1) = 3 \)
\[ f_1(x, y) = 2x, \quad f_1(-2, 1) = -4, \]
\[ f_2(x, y) = -2y, \quad f_2(-2, 1) = -2. \]
Tangent plane: \( z = 3 - 4(x + 2) - 2(y - 1) \), or \( 4x + 2y + z = -3 \)
Normal line: \( \frac{x - 1}{-4} = \frac{y - 2}{-2} = \frac{z + 1}{-3} \).

14. \( f(x, y) = \frac{x - y}{x + y} \), \( f(1, 1) = 0 \),
\[ f_1(x, y) = \frac{(x + y) - (x - y)}{(x + y)^2}, \quad f_1(1, 1) = \frac{1}{2}, \]
\[ f_2(x, y) = \frac{(x + y) - (x - y)}{(x + y)^2}, \quad f_2(1, 1) = \frac{1}{2}. \]
Tangent plane to \( z = f(x, y) \) at \((1, 1)\) has equation \( z = \frac{x - 1}{2} - \frac{y - 2}{2} \), or \( 2z = x - y \).
Normal line: \( 2(x - 1) = -2(y - 1) = -z \).
15. \( f(x, y) = \cos \frac{x}{y} \) \( f(x, 4) = \frac{1}{\sqrt{2}} \)

\( f_1(x, y) = -\frac{1}{y} \sin \frac{x}{y} \)

\( f_1(x, 4) = -\frac{1}{4\sqrt{2}} \)

\( f_2(x, y) = \frac{1}{y^2} \)

\( f_2(x, 4) = \frac{1}{16\sqrt{2}} \)

The tangent plane at \( x = \pi, y = 4 \) is

\[
z = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{4} (x - \pi) + \frac{\pi}{16} (y - 4) \right),
\]

or \( 4x - \pi y + 16\sqrt{2} = 16 \).

Normal line:

\[-4\sqrt{2}(x - \pi) + 16\sqrt{2}(y - 4) = (z - (1/\sqrt{2})).\]

16. \( f(x, y) = e^x y \) \( f_1(x, y) = ye^x, \ f_2(x, y) = xe^x \)

\( f(2, 0) = 1, \ f_1(2, 0) = 0, \ f_2(2, 0) = 2 \)

Tangent plane to \( z = xe^y \) at \( (2, 0) \) has equation \( z = 1 + 2y \).

Normal line: \( x = 2, \ y = 2 - 2z \).

17. \( f(x, y) = \frac{x}{x^2 + y^2} \)

\( f_1(x, y) = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \)

\( f_2(x, y) = -\frac{2xy}{x^2 + y^2} \)

\( f(1, 2) = \frac{1}{3}, \ f_1(1, 2) = \frac{3}{25}, \ f_2(1, 2) = -\frac{4}{25} \)

The tangent plane at \( x = 1, y = 2 \) is

\[
z = \frac{1}{3} + \frac{3}{25} (x - 1) - \frac{4}{25} (y - 2),
\]

or \( 3x - 4y - 25z = -10 \).

Normal line: \( \frac{x - 1}{3} = \frac{y - 2}{-4} = \frac{z - 5}{-125} \).

18. \( f(x, y) = ye^{-x^2} \) \( f_1(x, y) = -2xye^{-x^2}, \ f_2(x, y) = e^{-x^2} \)

\( f(0, 1) = 1, \ f_1(0, 1) = 0, \ f_2(0, 1) = 1 \)

Tangent plane to \( z = f(x, y) \) at \( (0, 1) \) has equation \( z = 1 + 1(y - 1), \ or \ z = y \).

Normal line: \( x = 0, \ y = z = 2 \).

19. \( f(x, y) = \ln(x^2 + y^2) \)

\( f_1(x, y) = \frac{2x}{x^2 + y^2} \)

\( f_1(x, -2) = \frac{2}{5} \)

\( f_2(x, y) = \frac{2y}{x^2 + y^2} \)

\( f_2(x, -2) = -\frac{4}{5} \)

The tangent plane at \( (1, -2, \ln 5) \) is

\[
z = \ln 5 + \frac{2}{5} (x - 1) - \frac{4}{5} (y + 2),
\]

or \( 2x - 4y - 5z = 10 - 5\ln 5 \).

Normal line: \( \frac{x - 1}{2/5} = \frac{y + 2}{-4/5} = \frac{z - \ln 5}{-1} \).

20. \( f(x, y) = \frac{2xy}{x^2 + y^2}, \ f(0, 2) = 0 \)

\( f_1(x, y) = \frac{(x^2 + y^2)2y - 2xy(2x)}{(x^2 + y^2)^2} = \frac{2y(y^2 - x^2)}{(x^2 + y^2)^2} \)

\( f_2(x, y) = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^2} \)

(by symmetry)

\( f_1(0, 2) = 1, \ f_2(0, 2) = 0 \)

Tangent plane at \( (0, 2) \): \( z = x \).

Normal line: \( z + x = 0, \ y = 2 \).

21. \( f(x, y) = \tan^{-1} \left( \frac{y}{x} \right), \ f(1, -1) = -\frac{\pi}{4} \)

\( f_1(x, y) = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \)

\( f_2(x, y) = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \)

\( f_1(1, -1) = f_2(1, -1) = \frac{1}{2} \)

The tangent plane is

\[
z = -\frac{\pi}{4} + \frac{1}{2} (x - 1) + \frac{1}{2} (y + 1), \ or \ z = -\frac{\pi}{4} + \frac{1}{2} (x + y) \).

Normal line: \( 2(x - 1) = 2(y + 1) = -z - \frac{\pi}{4} \).

22. \( f(x, y) = \sqrt{1 + x^2 y^2}, \ f(2, 1) = 3 \)

\( f_1(x, y) = \frac{3x^2y^2}{2\sqrt{1 + x^2 y^2}}, \ f_2(2, 1) = \frac{8}{3} \)

\( f_2(x, y) = \frac{2x^3y}{2\sqrt{1 + x^2 y^2}} \)

Tangent plane: \( z = 3 + 2(x - 2) + \frac{5}{2} (y - 1), \ or \ 6x + 8y - 3z = 11 \).

Normal line: \( \frac{x - 2}{8/3} = \frac{y - 1}{-1} = \frac{z - 3}{-1} \).

23. \( z = x^4 - 4xy^3 + 6y^2 - 2 \)

\[
\frac{\partial z}{\partial x} = 4x^3 - 4y^3 = 4(x - y)(x^2 + xy + y^2) \]

\[
\frac{\partial z}{\partial y} = -12xy^2 + 12y = 12y(1 - xy) \)

The tangent plane will be horizontal at points where both first partials are zero. Thus we require \( x = y \) and either \( y = 0 \) or \( xy = 1 \).

If \( x = y = 0 \), then \( x = 0 \).

If \( x = y \) and \( xy = 1 \), then \( x^2 = 1 \), so \( x = \pm 1 \).

The tangent plane is horizontal at the points \((0, 0), (1, 1), \) and \((-1, -1)\).

24. \( z = xye^{-(x^2+y^2)/2} \)

\[
\frac{\partial z}{\partial x} = yxe^{-(x^2+y^2)/2} - \frac{x}{2} y e^{-(x^2+y^2)/2} = y(1 - x^2) e^{-(x^2+y^2)/2} \]

\[
\frac{\partial z}{\partial y} = x(1 - y^2) e^{-(x^2+y^2)/2} \)

(by symmetry)

The tangent planes are horizontal at points where both of these first partials are zero, that is, points satisfying \( y(1 - x^2) = 0 \) and \( x(1 - y^2) = 0 \).
These points are (0, 0), (1, 1), (−1, −1), (1, −1) and (−1, 1).
At (0,0) the tangent plane is \( z = 0 \).
At (1, 1) and (−1, −1) the tangent plane is \( z = 1/e \).
At (1, −1) and (−1, 1) the tangent plane is \( z = −1/e \).

25. If \( z = xe^y \), then \( \frac{\partial z}{\partial x} = e^y \) and \( \frac{\partial z}{\partial y} = xe^y \).

Thus \( \frac{\partial z}{\partial x} = e^y \) and \( \frac{\partial z}{\partial y} = xe^y \).

26. \( z = \frac{x + y}{x − y} \)
\( \frac{\partial z}{\partial x} = \frac{(x − y)(1) − (x + y)(1)}{(x − y)^2} = \frac{−2y}{(x − y)^2} \)
\( \frac{\partial z}{\partial y} = \frac{(x − y)(1) − (x + y)(1)}{(x − y)^2} = \frac{2x}{(x − y)^2} \).

Therefore \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{−2xy}{(x − y)^2} + \frac{2xy}{(x − y)^2} = 0 \).

27. If \( z = \sqrt{x^2 + x^2} \), then \( \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \) and \( \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \).

Thus \( \frac{x}{\partial x} \frac{\partial z}{\partial x} + \frac{y}{\partial y} \frac{\partial z}{\partial y} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = z \).

28. \( w = x^2 + yz \), \( \frac{\partial w}{\partial x} = 2x \), \( \frac{\partial w}{\partial y} = z \), \( \frac{\partial w}{\partial z} = y \).

Therefore \( \frac{x}{\partial x} \frac{\partial w}{\partial x} + \frac{y}{\partial y} \frac{\partial w}{\partial y} + \frac{z}{\partial z} \frac{\partial w}{\partial z} = 2x^2 + yz + yz = 2(x^2 + yz) = 2w \).

29. If \( w = \frac{1}{x^2 + y^2} \), then \( \frac{\partial w}{\partial x} = -\frac{2x}{(x^2 + y^2)^2} \), \( \frac{\partial w}{\partial y} = -\frac{2y}{(x^2 + y^2)^2} \).

Thus \( \frac{x}{\partial x} \frac{\partial w}{\partial x} + \frac{y}{\partial y} \frac{\partial w}{\partial y} + \frac{z}{\partial z} \frac{\partial w}{\partial z} = -2 \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = -2w \).

30. \( z = f(x^2 + y^2) \),
\( \frac{\partial z}{\partial x} = f'(x^2 + y^2)(2x) \), \( \frac{\partial z}{\partial y} = f'(x^2 + y^2)(2y) \).

Thus \( \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = 2x y f'(x^2 + y^2) - 2x y f'(x^2 + y^2) = 0 \).

31. \( z = f(x^2 − y^2) \),
\( \frac{\partial z}{\partial x} = f'(x^2 − y^2)(2x) \), \( \frac{\partial z}{\partial y} = f'(x^2 − y^2)(−2y) \).

Thus \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = (2xy − 2xy)f'(x^2 − y^2) = 0 \).

32. \( f_1(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) − f(x, y, z)}{h} \)
\( f_2(x, y, z) = \lim_{k \to 0} \frac{f(x, y + k, z) − f(x, y, z)}{k} \)
\( f_3(x, y, z) = \lim_{\ell \to 0} \frac{f(x, y, z + \ell) − f(x, y, z)}{\ell} \).

33. At \((a, b, c, f(a, b, c))\) the graph of \( w = f(x, y, z) \) has tangent hyperplane
\( w = f(a, b, c) + f_1(a, b, c)(x − a) + f_2(a, b, c)(y − b) + f_3(a, b, c)(z − c) \).

34. If \( Q = (X, Y, Z) \) is the point on the surface \( z = x^2 + y^2 \) that is closest to \( P = (1, 1, 0) \), then
\( \overrightarrow{PQ} = (X − 1) + (Y − 1)j + Zk \)
must be normal to the surface at \( Q \), and hence must be parallel to \( n = 2Xj + 2Yj − k \). Hence \( \overrightarrow{PQ} = tn \) for some real number \( t \), so
\( X − 1 = 2tX \), \( Y − 1 = 2tY \), \( Z = −t \).

Thus \( X = Y = \frac{1}{1−2t} \), and, since \( Z = X^2 + Y^2 \), we must have
\( −t = \frac{2}{(1−2t)^2} \).

Evidently this equation is satisfied by \( t = \frac{1}{2} \). Since the left and right sides of the equation have graphs similar to those in Figure 12.18(b) (in the text), the equation has only this one real solution. Hence \( X = Y = \frac{1}{2} \), and so \( Z = \frac{7}{2} \).

The distance from \((1, 1, 0)\) to \( z = x^2 \) is the distance from \((1, 1, 0)\) to \((\frac{1}{2}, \frac{1}{2}, \frac{7}{2}) \), which is \( \sqrt{3}/2 \) units.

35. If \( Q = (X, Y, Z) \) is the point on the surface \( z = x^2 + y^2 \) that is closest to \( P = (0, 0, 1) \), then
\( \overrightarrow{PQ} = Xj + Yj + (Z − 1)k \)
must be normal to the surface at \( Q \), and hence must be parallel to \( n = 2Xj + 4Yj − k \). Hence \( \overrightarrow{PQ} = tn \) for some real number \( t \), so
\( X = 2tX \), \( Y = 4tY \), \( Z − 1 = −t \).
If \( X \neq 0 \), then \( t = 1/2 \), so \( Y = 0, Z = 1/2 \), and \( X = \sqrt{Z} = 1/\sqrt{2} \). The distance from \((1/\sqrt{2}, 0, 1/2)\) to \((0, 0, 1)\) is \(\sqrt{3}/2\) units.

If \( Y \neq 0 \), then \( t = 1/4 \), so \( X = 0, Z = 3/4 \), and \( Y = \sqrt{Z/2} = \sqrt{3}/8 \). The distance from \((0, \sqrt{3}/8, 3/4)\) to \((0, 0, 1)\) is \(\sqrt{7}/4\) units.

If \( X = Y = 0 \), then \( Z = 0 \) (and \( t = 1 \)). The distance from \((0, 0, 0)\) to \((0, 0, 1)\) is 1 unit.

Since \( \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{2} < 1 \), the closest point to \((0, 0, 1)\) on \( z = x^2 + 2y^2 \) is \((0, \sqrt{3}/8, 3/4)\), and the distance from \((0, 0, 1)\) to that surface is \(\sqrt{7}/4\) units.

36. \( f(x, y) = \frac{2xy}{x^2 + y^2} \) if \((x, y) \neq (0, 0), \quad f(0, 0) = 0 \)

\[ f_1(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0 \]

Thus \( f_1(0, 0) \) exists even though \( f \) is not continuous at \((0, 0)\) (as shown in Example 2 of Section 3.2).

37. \( f(x, y) = \begin{cases} (x^3 + y) \sin \frac{1}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \)

\[ f_1(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0 \]

Thus \( f_1(0, 0) = 0 \) and \( f_2(0, 0) \) both exist even though \( f \) is not continuous at \((0, 0)\).

38. If \((x, y) \neq (0, 0)\), then

\[ f_1(x, y) = 3x^2 \sin \frac{1}{x^2 + y^2} - \frac{(x^3 + y)2x}{(x^2 + y^2)^2} \cos \frac{1}{x^2 + y^2} \]

The first term on the right \(\to 0\) as \((x, y) \to (0, 0)\), but the second term has no limit at \((0, 0)\). (It is 0 along \(x \to 0\), but along \(x = y\) it is

\[ \lim_{y \to 0} \frac{4x^4 + 2x^2 \cos \frac{1}{2x^2}}{4x^4} = \frac{1}{2} \left(1 + \frac{1}{x^2}\right) \cos \frac{1}{2x^2} \]

which has no limit as \(x \to 0\). Thus \( f_1(x, y) \) has no limit at \((0, 0)\) and is not continuous there.

39. \( f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) , \\ 0 & \text{if } (x, y) = (0, 0) . \end{cases} \)

If \((x, y) \neq (0, 0)\), then

\[ f_1(x, y) = \frac{(x^2 + y^2)(x^3 - y^3)2x - (x^3 - y^3)2x}{(x^2 + y^2)^2} \]

\[ = \frac{4x^4 + 2x^2 y^2 + 2x^2 y^2}{(x^2 + y^2)^2} \]

\[ f_2(x, y) = \frac{(x^2 + y^2)(-3y^2) - (x^3 - y^3)2y}{(x^2 + y^2)^2} \]

\[ = \frac{-y^4 + 3x^2 y^2 + 2x^2 y}{(x^2 + y^2)^2} \]

At \((0, 0)\),

\[ f_1(0, 0) = \lim_{h \to 0} \frac{h^3}{h} = 1, \quad f_2(0, 0) = \lim_{k \to 0} \frac{-k^3}{k} = -1 . \]

Neither \( f_1 \) nor \( f_2 \) has a limit at \((0, 0)\). (The limits along \(x = 0\) and \(y = 0\) are different in each case), so neither function is continuous at \((0, 0)\). However, \( f \) is continuous at \((0, 0)\) because

\[ |f(x, y)| \leq \frac{x^3}{x^2 + y^2} + \frac{y^3}{x^2 + y^2} \leq |x| + |y| \]

which \(\to 0\) as \((x, y) \to (0, 0)\).

40. \( f(x, y, z) = \begin{cases} \frac{x^2 y^2 z}{x^4 + y^4 + z^4} & \text{if } (x, y, z) \neq (0, 0, 0), \\ 0 & \text{if } (x, y, z) = (0, 0, 0) . \end{cases} \)

By symmetry we have

\[ f_3(0, 0, 0) = f_1(0, 0, 0) = \lim_{h \to 0} \frac{0}{h} = 0 . \]

Also,

\[ f_2(0, 0, 0) = \lim_{k \to 0} \frac{0}{k} = 0 . \]

\( f \) is not continuous at \((0, 0, 0)\); it has different limits as \((x, y, z) \to (0, 0, 0)\) along \(x = 0\) and along \(x = y = z\). None of \( f_1, f_2, \) and \( f_3 \) is continuous at \((0, 0, 0)\) either.

For example,

\[ f_3(x, y, z) = \frac{(y^4 + z^4 - 3x^4)2y^2}{(x^4 + y^4 + z^4)^2} \]

which has no limit as \((x, y, z) \to (0, 0, 0)\) along the line \(x = y = z\).

Section 12.4 Higher-Order Derivatives

(page 702)

1. \( z = x^2(1 + y^2) \)

\[ \frac{\partial z}{\partial x} = 2x(1 + y^2), \quad \frac{\partial z}{\partial y} = 2x^2y, \]

\[ \frac{\partial^2 z}{\partial x^2} = 2(1 + y^2), \quad \frac{\partial^2 z}{\partial y^2} = 2x^2, \]

\[ \frac{\partial^2 z}{\partial y \partial x} = 4xy = \frac{\partial^2 z}{\partial x \partial y} . \]
2. \( f(x, y) = x^2 + y^2 \)  
\( f_1(x, y) = 2x \),  
\( f_2(x, y) = 2y \),  
\( f_{11}(x, y) = f_{22}(x, y) = 2 \),  
\( f_{12}(x, y) = f_{21}(x, y) = 0 \).

3. \( w = x^3 y^3 z \).
\( \frac{\partial w}{\partial x} = 3x^2 y^3 z^3 \),  
\( \frac{\partial w}{\partial y} = 3x^3 y^2 z^3 \),  
\( \frac{\partial w}{\partial z} = 3x^3 y^3 z^2 \),  
\( \frac{\partial^2 w}{\partial x^2} = 6x^3 y^3 z^3 \),  
\( \frac{\partial^2 w}{\partial y^2} = 6x^3 y^3 z^3 \),  
\( \frac{\partial^2 w}{\partial z^2} = 6x^3 y^3 z^3 \).  
\( \frac{\partial^2 w}{\partial x \partial y} = 9x^2 y^2 z^3 \),  
\( \frac{\partial^2 w}{\partial x \partial z} = 9x^2 y^2 z^3 \).  
\( \frac{\partial^2 w}{\partial y \partial z} = 9x^2 y^2 z^3 \),  
\( \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial x} \).

4. \( z = \sqrt{3x^2 + y^2} \).
\( \frac{\partial z}{\partial x} = \frac{3x}{\sqrt{3x^2 + y^2}} \),  
\( \frac{\partial z}{\partial y} = \frac{y}{\sqrt{3x^2 + y^2}} \).  
\( \frac{\partial^2 z}{\partial x^2} = \frac{3x^2 + y^2 - 3x^2}{(3x^2 + y^2)^{3/2}} = \frac{y}{(3x^2 + y^2)^{3/2}} \),  
\( \frac{\partial^2 z}{\partial y^2} = \frac{3x^2 + y^2 - y^2}{(3x^2 + y^2)^{3/2}} = \frac{3x}{(3x^2 + y^2)^{3/2}} \),  
\( \frac{\partial^2 z}{\partial y \partial x} = \frac{3y}{(3x^2 + y^2)^{3/2}} \).

5. \( z = x e^y - y e^x \).
\( \frac{\partial z}{\partial x} = e^y - y e^x \),  
\( \frac{\partial z}{\partial y} = x e^y - e^x \).  
\( \frac{\partial^2 z}{\partial x^2} = -e^y \),  
\( \frac{\partial^2 z}{\partial y^2} = x e^x \),  
\( \frac{\partial^2 z}{\partial y \partial x} = e^y - e^x \).

6. \( f(x, y) = \ln(1 + \sin(xy)) \).
\( f_1(x, y) = \frac{y \cos(xy)}{1 + \sin(xy)} \),  
\( f_2(x, y) = \frac{x \cos(xy)}{1 + \sin(xy)} \).
\( f_{11}(x, y) = \frac{(1 + \sin(xy))(-y^2 \sin(xy)) - (y \cos(xy))(y \cos(xy))}{(1 + \sin(xy))^2} \).
\( f_{12}(x, y) = \frac{-xy \cos(xy)}{1 + \sin(xy)} \).
\( f_{22}(x, y) = \frac{-xy \cos(xy)}{1 + \sin(xy)} \) (by symmetry)
\( f_{12}(x, y) = \frac{\cos(xy) - xy}{1 + \sin(xy)} = f_{21}(x, y) \).

7. A function \( f(x, y, z) \) of three variables can have \( 3^3 = 27 \) partial derivatives of order 3. Of these, ten can have different values, namely \( f_{111}, f_{222}, f_{333}, f_{112}, f_{121}, f_{223}, f_{232}, f_{311}, f_{321}, \) and \( f_{123} \).
\( f_{113} = f_{313} = \frac{\partial}{\partial x} \left( -x^3 e^{x^2} \cos(xz) \right) \)
\( = -3x^2 e^{x^2} \cos(xz) + x^2 \sin xz \).

8. \( f(x, y) = A(x^2 - y^2) + Bx y \).
\( f_1 = 2Ax + By \),  
\( f_2 = -2Ay + Bx \).
\( f_{11} = 2A \),  
\( f_{22} = -2A \).
Thus \( f_{11} + f_{22} = 0 \) and \( f \) is harmonic.

9. \( f(x, y) = x e^y - y^3 \).
\( f_1(x, y) = 6e^y \),  
\( f_2(x, y) = 3x e^y - 3y^2 \).
\( f_{11}(x, y) = 6e^y \).
Thus \( f_{11} + f_{22} = 0 \) and \( f \) is harmonic.
Also \( g(x, y) = x^3 - 3xy^2 \) is harmonic.

10. \( f(x, y) = \frac{x}{x^2 + y^2} \).
\( f_1(x, y) = -\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \)
\( f_2(x, y) = -\frac{2xy}{(x^2 + y^2)^2} \).
\( f_{11}(x, y) = \frac{(x^2 + y^2)^2(2x) - (x^2 - y^2)^2(2y)}{(x^2 + y^2)^4} \)
\( = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3} \),  
\( f_{22}(x, y) = \frac{(x^2 + y^2)^2(2y) - 2xy(2x^2 + y^2)(2y)}{(x^2 + y^2)^4} \)
\( = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3} \).
Evidently \( f_{11}(x, y) + f_{22}(x, y) = 0 \) for \( (x, y) \neq (0, 0) \).
Hence \( f \) is harmonic except at the origin.

11. \( f(x, y) = \ln(x^2 + y^2) \).
\( f_1 = \frac{2x}{x^2 + y^2} \),  
\( f_2 = \frac{2y}{x^2 + y^2} \).
\( f_{11} = \frac{(x^2 + y^2)(2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} \),  
\( f_{22} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \) (by symmetry)
Thus \( f_{11} + f_{22} = 0 \) (everywhere except at the origin), and \( f \) is harmonic.
12. \( f(x, y) = \tan^{-1} \left( \frac{y}{x} \right), \quad (x \neq 0). \)
\[
f'_1(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2},
\]
\[
f'_2(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right) = \frac{x}{x^2 + y^2},
\]
\[
f_{11} = \frac{2xy}{(x^2 + y^2)^2}, \quad f_{22} = -\frac{2xy}{(x^2 + y^2)^2}.
\]
Thus \( f'_{11} + f'_{22} = 0 \) and \( f \) is harmonic.

13. \( w = e^{x+4y} \sin(5z), \)
\( w_1 = 3w, \quad w_2 = 4w, \quad w_{11} = 9w, \quad w_{22} = 16w, \)
\( w_3 = 5e^{3x+4y} \cos(5z), \quad w_{33} = -25w. \)
Thus \( w_{11} + w_{22} + w_{33} = (9 + 16 - 25)w = 0, \) and \( w \) is harmonic in 3-space.

14. Let \( g(x, y, z) = zf(x, y). \) Then
\[
g'_1(x, y, z) = z f'_1(x, y), \quad g'_{11}(x, y, z) = z f'_{11}(x, y),
\]
\[
g'_2(x, y, z) = z f'_2(x, y), \quad g'_{22}(x, y, z) = z f'_{22}(x, y),
\]
\[
g'_3(x, y, z) = f(x, y), \quad g'_{33}(x, y, z) = 0.
\]
Thus \( g'_{11} + g'_{22} + g'_{33} = z(f'_{11} + f'_{22}) = 0 \) and \( g \) is harmonic because \( f \) is harmonic. This proves (a). The proofs of (b) and (c) are similar.

If \( h(x, y, z) = f(ax + by, cz), \) then \( h'_{11} = a^2f'_{11}, \)
\( h_{22} = b^2f'_{22} \) and \( h_{33} = c^2f'_{33}. \) If \( a^2 + b^2 = c^2 \) and \( f \) is harmonic then
\[
h_{11} + h_{22} + h_{33} = c^2(f'_{11} + f'_{22}) = 0,
\]
so \( h \) is harmonic.

15. Since \( \frac{\partial u}{\partial x} = \frac{\partial y}{\partial y} \cdot \frac{\partial u}{\partial y}, \) and the partials of \( u \) are continuous, we have
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2}.
\]
Thus \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \) and \( u \) is harmonic. The proof that \( v \) is harmonic is similar.

16. Let
\[
f(x, y) = \begin{cases} 2xy & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}
\]
For \( (x, y) \neq (0, 0), \) we have
\[
f_1(x, y) = \frac{(x^2 + y^2)2y - 2xy(2x)}{(x^2 + y^2)^2} = \frac{2y(y^2 - x^2)}{(x^2 + y^2)^2},
\]
\[
f_2(x, y) = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^2} \quad \text{(by symmetry)}.
\]
Let \( F(x, y) = (x^2 - y^2)f(x, y). \) Then we calculate
\[
F'_1(x, y) = 2xf(x, y) + (x^2 - y^2)f'_1(x, y) = 2xf(x, y) - 2y(y^2 - x^2)
\]
\[
\frac{(x^2 + y^2)^3}{(x^2 + y^2)^2},
\]
\[
F'_2(x, y) = -2yf(x, y) + (x^2 - y^2)f'_2(x, y) = -2yf(x, y) + 2x(x^2 - y^2)
\]
\[
\frac{(x^2 + y^2)^3}{(x^2 + y^2)^2}.
\]
\[
F_{12}(x, y) = \frac{2(x^2 + 9x^2 y^2 - 9x^2 y^4 - y^6)}{(x^2 + y^2)^3} = F_{21}(x, y).
\]

For the values at \( (0, 0) \) we revert to the definition of derivative to calculate the partials:
\[
F'_1(0, 0) = \lim_{h \to 0} \frac{F(h, 0) - F(0, 0)}{h} = 0 = F_{21}(0, 0)
\]
\[
F'_1(0, k) = -2k^4, \quad F'_2(0, k) = 2k^4.
\]
\[
F_{21}(0, 0) = 0 = \lim_{h \to 0} \frac{F(0, k) - F(0, 0)}{h} = \lim_{h \to 0} \frac{2h^4}{h} = 2
\]
This does not contradict Theorem 1 since the partials \( F'_{11} \) and \( F'_{22} \) are not continuous at \( (0, 0). \) (Observe, for instance, that \( F'_{12}(x, x) = 0, \) while \( F_{12}(x, 0) = 2 \) for \( x \neq 0. \))

17. \( u(x, t) = t^{-1/2} e^{x^2/4t} \)
\[
\frac{\partial u}{\partial t} = \left( \frac{1}{2} x^2 - \frac{1}{4} t \right) e^{x^2/4t},
\]
\[
\frac{\partial u}{\partial x} = -\frac{1}{2} x e^{x^2/4t},
\]
\[
\frac{\partial^2 u}{\partial x^2} = \left( \frac{1}{2} x^2 + \frac{1}{4} t \right) e^{x^2/4t}.
\]

18. \( u(x, y, t) = t^{-1} e^{-(x^2+y^2)/4t} \)
\[
\frac{\partial u}{\partial t} = \frac{1}{4t^2} e^{-(x^2+y^2)/4t} x^2 + \frac{1}{4t^2} e^{-(x^2+y^2)/4t} y^2
\]
\[
\frac{\partial u}{\partial x} = -\frac{x}{2t^2} e^{-(x^2+y^2)/4t},
\]
\[
\frac{\partial^2 u}{\partial x^2} = -\frac{1}{4t^2} e^{-(x^2+y^2)/4t} x^2 + \frac{1}{2t^2} e^{-(x^2+y^2)/4t}
\]
\[
\frac{\partial^2 u}{\partial y^2} = -\frac{1}{4t^2} e^{-(x^2+y^2)/4t} y^2 + \frac{1}{2t^2} e^{-(x^2+y^2)/4t},
\]
\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2t^2} e^{-(x^2+y^2)/4t}.
\]
Thus \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \)

19. For \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \) the solution is
\[
u(x, y, z, t) = t^{-3/2} e^{-(x^2+y^2+z^2)/4t},
\]
which is verified similarly to the previous Exercise.
20. If \(u(x, y)\) is biharmonic \(\Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\) is harmonic

\[
\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0
\]

\[
\Rightarrow \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0
\]

by the equality of mixed partials.

21. If \(u(x, y) = x^2 - 3xy^2\), then

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(4x^3 - 6xy^2\right) = 12x^2 - 6y^2 \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} (-6x^2y) = -6x^2 \\
\frac{\partial^4 u}{\partial x^4} &= \frac{\partial}{\partial x} (24x) = 24 \\
\frac{\partial^4 u}{\partial x^2 \partial y^2} &= \frac{\partial}{\partial x} (-12x) = -12 \\
\frac{\partial^4 u}{\partial y^4} &= 0
\end{align*}
\]

Thus \(u\) is biharmonic.

22. If \(u\) is harmonic, then \(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\). If

\[
v(x, y) = xu(x, y),
\]

then

\[
\begin{align*}
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x} \left(x \frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} \\
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y} \left(x \frac{\partial u}{\partial y}\right) = x \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 2 \frac{\partial u}{\partial x} + 2 \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial y^2} = 2 \frac{\partial u}{\partial x}.
\end{align*}
\]

Since \(u\) is harmonic, so is \(\partial u/\partial x\):

\[
\begin{align*}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) &= 2 \frac{\partial u}{\partial x}(0) = 0.
\end{align*}
\]

Thus \(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\) is harmonic, and so \(v\) is biharmonic. The proof that \(u(x, y) = yu(x, y)\) is biharmonic is similar.

23. By Example 3, \(e^x \sin y\) is harmonic. Therefore \(xe^x \sin y\) is biharmonic by Exercise 22.

24. By Exercise 11, \(\ln(x^2 + y^2)\) is harmonic (except at the origin). Therefore \(y \ln(x^2 + y^2)\) is biharmonic by Exercise 22.

25. By Exercise 10, \(\frac{x}{x^2 + y^2}\) is harmonic (except at the origin). Therefore \(\frac{xy}{x^2 + y^2}\) is biharmonic by Exercise 22.

26. If \(u(x, y, z)\) is biharmonic \(\Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\) is harmonic

\[
\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0
\]

\[
\Rightarrow \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4} + 2 \left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2}\right) = 0
\]

by the equality of mixed partials.

If \(u(x, y, z)\) is harmonic then the functions \(xu(x, y, z), yu(x, y, z), z u(x, y, z)\) are all biharmonic. The proof is almost identical to that given in Exercise 22.

27. \(f := \frac{x^2+y^2}{2}/\sqrt{x^2+y^2} + \frac{xy}{2}/\sqrt{x^2+y^2} + \frac{y^2}{2}/\sqrt{x^2+y^2}\)

\(= \) simplify(diff(f,x,y)+2*diff(f,x,y)+diff(f,y));

\[0\]

Section 12.5 The Chain Rule (page 711)

1. If \(w = f(x, y, z)\) where \(x = g(s, t), y = h(s, t),\) and \(z = k(s, t),\) then

\[
\frac{\partial w}{\partial t} = f_1(x, y, z)g_2(s, t) + f_2(x, y, z)h_2(s, t) + f_3(x, y, z)k_2(s, t).
\]

2. If \(w = f(x, y, z)\) where \(x = g(s), y = h(s, t)\) and \(z = k(t),\) then

\[
\frac{\partial w}{\partial t} = f_2(x, y, z)h_2(s, t) + f_3(x, y, z)k'(t).
\]

3. If \(z = g(x, y)\) where \(y = f(x)\) and \(x = h(u, v),\) then

\[
\frac{\partial z}{\partial u} = g_1(x, y)h_2(u, v) + g_2(x, y)f'(x)h_1(u, v).
\]

4. If \(w = f(x, y)\) where \(x = g(r, s), y = h(r, t), r = k(s, t)\) and \(s = m(t),\) then

\[
\frac{dw}{dT} = f_1(x, y)\left[g_1(r, s)\left(k_1(s, t)m'(t) + k_2(s, t)\right)\right] + f_2(x, y)\left[h_1(r, t)\left(k_1(s, t)m'(t) + k_2(s, t)\right) + h_2(r, t)\right] + k_2(s, t).
\]