27. $f(2) = 4, \quad f'(2) = -1, \quad 0 \le f''(x) \le \frac{1}{x}$ if $x > 0$.
$f(3) \approx f(2) + f'(2)(3-2) = 4 - 1 = 3$.
$f''(x) \ge 0 \Rightarrow$ error $\ge 0 \Rightarrow f(3) \ge 3$.
$|f''(x)| \le \frac{1}{x} \le \frac{1}{2}$ if $2 \le x \le 3$, so $|\text{Error}| \le \frac{1}{4}(3-2)^2$.
Thus $3 \le f(3) \le 3\frac{1}{4}$

28. The linearization of $f(x)$ at $x = 2$ is

$$L(x) = f(2) + f'(2)(x-2) = 4 - (x-2).$$

Thus $L(3) = 3$. Also, since $1/(2x) \le f''(x) \le 1/x$ for $x > 0$, we have for $2 \le x \le 3$, $(1/6) \le f''(x) \le (1/2)$. Thus

$$3 + \frac{1}{2}\left(\frac{1}{6}\right)(3-2)^2 \le f(3) \le 3 + \frac{1}{2}\left(\frac{1}{2}\right)(3-2)^2.$$

The best approximation for $f(3)$ is the midpoint of this interval: $f(3) \approx 3\frac{1}{6}$.

29. The linearization of $g(x)$ at $x = 2$ is

$$L(x) = g(2) + g'(2)(x-2) = 1 + 2(x-2).$$

Thus $L(1.8) = 0.6$.
If $|g''(x)| \le 1 + (x-2)^2$ for $x > 0$, then $|g''(x)| < 1 + (-0.2)^2 = 1.04$ for $1.8 \le x \le 2$. Hence $g(1.8) \approx 0.6$ with $|\text{error}| < \frac{1}{2}(1.04)(1.8-2)^2 = 0.0208$.

30. If $f(\theta) = \sin\theta$, then $f'(\theta) = \cos\theta$ and $f''(\theta) = -\sin\theta$. Since $f(0) = 0$ and $f'(0) = 1$, the linearization of f at $\theta = 0$ is $L(\theta) = 0 + 1(\theta - 0) = \theta$.
If $0 \le t \le \theta$, then $f''(t) \le 0$, so $0 \le \sin\theta \le \theta$.
If $0 \ge t \ge \theta$, then $f''(t) \ge 0$, so $0 \ge \sin\theta \ge \theta$.
In either case, $|\sin t| \le |\sin\theta| \le |\theta|$ if t is between 0 and θ. Thus the error $E(\theta)$ in the approximation $\sin\theta \approx \theta$ satisfies

$$|E(\theta)| \le \frac{|\theta|}{2}|\theta|^2 = \frac{|\theta|^3}{2}.$$

If $|\theta| \le 17° = 17\pi/180$, then

$$\frac{|E(\theta)|}{|\theta|} \le \frac{1}{2}\left(\frac{17\pi}{180}\right)^2 \approx 0.044.$$

Thus the percentage error is less than 5%.

31. $V = \frac{4}{3}\pi r^3 \Rightarrow \Delta V \approx 4\pi r^2 \Delta r$
If $r = 20.00$ and $\Delta r = 0.20$, then
$\Delta V \approx 4\pi(20.00)^2(0.20) \approx 1005$.
The volume has increased by about 1005 cm^2.

Section 4.10 Taylor Polynomials (page 280)

1. If $f(x) = e^{-x}$, then $f^{(k)}(x) = (-1)^k e^{-x}$, so $f^{(k)}(0) = (-1)^k$. Thus

$$P_4(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}.$$

2. If $f(x) = \cos x$, then $f'(x) = -\sin x$, $f''(x) = -\cos x$, and $f'''(x) = \sin x$. In particular, $f(\pi/4) = f'''(\pi/4) = 1/\sqrt{2}$ and $f'(\pi/4) = f''(\pi/4) = -1/\sqrt{2}$. Thus

$$P_3(x) = \frac{1}{\sqrt{2}}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right].$$

3.

$$\begin{aligned} f(x) &= \ln x & f(2) &= \ln 2 \\ f'(x) &= \frac{1}{x} & f'(2) &= \frac{1}{2} \\ f''(x) &= \frac{-1}{x^2} & f''(2) &= \frac{-1}{4} \\ f'''(x) &= \frac{2}{x^3} & f'''(2) &= \frac{2}{8} \\ f^{(4)}(x) &= \frac{-6}{x^4} & f^{(4)}(2) &= \frac{-6}{16} \end{aligned}$$

Thus

$$P_4(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4.$$

4.

$$\begin{aligned} f(x) &= \sec x & f(0) &= 1 \\ f'(x) &= \sec x \tan x & f'(0) &= 0 \\ f''(x) &= 2\sec^3 x - \sec x & f''(0) &= 1 \\ f'''(x) &= (6\sec^2 x - 1)\sec x \tan x & f'''(0) &= 0 \end{aligned}$$

Thus $P_3(x) = 1 + (x^2/2)$.

5.

$$\begin{aligned} f(x) &= x^{1/2} & f(4) &= 2 \\ f'(x) &= \frac{1}{2}x^{-1/2} & f'(4) &= \frac{1}{4} \\ f''(x) &= \frac{-1}{4}x^{-3/2} & f''(4) &= \frac{-1}{32} \\ f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(4) &= \frac{3}{256} \end{aligned}$$

Thus

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3.$$

6.
$$\begin{aligned} f(x) &= (1-x)^{-1} & f(0) &= 1 \\ f'(x) &= (1-x)^{-2} & f'(0) &= 1 \\ f''(x) &= 2(1-x)^{-3} & f''(0) &= 2 \\ f'''(x) &= 3!(1-x)^{-4} & f'''(0) &= 3! \\ &\vdots & &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-(n+1)} & f^{(n)}(0) &= n! \end{aligned}$$

Thus

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n.$$

7.
$$\begin{aligned} f(x) &= \frac{1}{2+x} & f(1) &= \frac{1}{3} \\ f'(x) &= \frac{-1}{(2+x)^2} & f'(1) &= \frac{-1}{9} \\ f''(x) &= \frac{2!}{(2+x)^3} & f''(1) &= \frac{2!}{27} \\ f'''(x) &= \frac{-3!}{(2+x)^4} & f'''(1) &= \frac{-3!}{3^4} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \frac{(-1)^n n!}{(2+x)^{n+1}} & f^{(n)}(1) &= \frac{(-1)^n n!}{3^{n+1}} \end{aligned}$$

Thus

$$P_n(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \cdots + \frac{(-1)^n}{3^{n+1}}(x-1)^n.$$

8.
$$\begin{aligned} f(x) &= \sin(2x) & f(\pi/2) &= 0 \\ f'(x) &= 2\cos(2x) & f'(\pi/2) &= -2 \\ f''(x) &= -2^2\sin(2x) & f''(\pi/2) &= 0 \\ f'''(x) &= -2^3\cos(2x) & f'''(\pi/2) &= 2^3 \\ f^{(4)}(x) &= 2^4\sin(2x) = 2^4 f(x) & f^{(4)}(\pi/2) &= 0 \\ f^{(5)}(x) &= 2^4 f'(x) & f^{(5)}(\pi/2) &= -2^5 \\ &\vdots & &\vdots \end{aligned}$$

Evidently $f^{(2n)}(\pi/2) = 0$ and $f^{(2n-1)}(\pi/2) = (-1)^n 2^{2n-1}$. Thus

$$P_{2n-1}(x) = -2\left(x-\frac{\pi}{2}\right) + \frac{2^3}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{2^5}{5!}\left(x-\frac{\pi}{2}\right)^5 + \cdots + (-1)^n\frac{2^{2n-1}}{(2n-1)!}\left(x-\frac{\pi}{2}\right)^{2n-1}.$$

9. $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$,
$f''(x) = -\frac{2}{9}x^{-5/3}$, $f'''(x) = \frac{10}{27}x^{-8/3}$.

$$\begin{aligned} a = 8: \quad f(x) &\approx f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{9\times 32}(x-8)^2 \end{aligned}$$

$9^{1/2} \approx 2 + \frac{1}{12} - \frac{1}{288} \approx 2.07986$

Error $= \dfrac{f'''(c)}{3!}(9-8)^3 = \dfrac{10}{27\times 6}\dfrac{1}{X^{8/3}}$ for some c in $[8, 9]$.
For $8 \le c \le 9$ we have $c^{8/3} \ge 8^{8/3} = 2^8 = 256$ so

$$0 < \text{Error} \le \frac{5}{81 \times 256} < 0.000241.$$

Thus $2.07986 < 9^{1/3} < 2.08010$.

10. Since $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$ and $f'''(x) = \frac{3}{8}x^{-5/2}$. Hence,

$$\begin{aligned} \sqrt{61} &\approx f(64) + f'(64)(61-64) + \frac{1}{2}f''(64)(61-64)^2 \\ &= 8 + \frac{1}{16}(-3) - \frac{1}{2}\left(\frac{1}{2048}\right)(-3)^2 \approx 7.8103027. \end{aligned}$$

The error is $R_2 = R_2(f; 64, 61) = \dfrac{f'''(c)}{3!}(61-64)^3$ for some c between 61 and 64. Clearly $R_2 < 0$. If $t \ge 49$, and in particular $61 \le t \le 64$, then

$$|f'''(t)| \le \tfrac{3}{8}(49)^{-5/2} = 0.0000223 = K.$$

Hence,

$$|R_2| \le \frac{K}{3!}|61-64|^3 = 0.0001004.$$

Since $R_2 < 0$, therefore,

$$\begin{aligned} 7.8103027 - 0.0001004 < \sqrt{61} &< 7.8103027 \\ 7.8102023 < \sqrt{61} &< 7.8103027. \end{aligned}$$

11. $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$,
$f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-6}{x^4}$.
$a = 1: f(x) \approx 1 - (x-1) + \frac{2}{2}(x-1)^2$
$\frac{1}{1.02} \approx 1 - (0.02) + (0.02)^2 = 0.9804$.
Error $= \dfrac{f'''(c)}{3!}(0.02)^3 = -\dfrac{1}{X^4}(0.02)^3$ where $1 \le c \le 1.02$.
Therefore, $-(0.02)^3 \le \dfrac{1}{1.02} - 0.9804 < 0$,
i.e., $0.980392 \le \dfrac{1}{1.02} < 0.980400$.

12. Since $f(x) = \tan^{-1} x$, then

$$f'(x) = \frac{1}{1+x^2},\ f''(x) = \frac{-2x}{(1+x^2)^2},\ f'''(x) = \frac{-2+6x^2}{(1+x^2)^3}.$$

Hence,

$$\begin{aligned}\tan^{-1}(0.97) &\approx f(1) + f'(1)(0.97-1) + \tfrac{1}{2}f''(1)(0.97-1)^2\\ &= \frac{\pi}{4} + \frac{1}{2}(-0.03) + \left(-\frac{1}{4}\right)(-0.03)^2\\ &= 0.7701731.\end{aligned}$$

The error is $R_2 = \dfrac{f'''(c)}{3!}(-0.03)^3$ for some c between 0.97 and 1. Note that $R_2 < 0$. If $0.97 \le t \le 1$, then

$$|f'''(t)| \le f'''(1) = \frac{-2+6}{(1.97)^3} < 0.5232 = K.$$

Hence,

$$|R_2| \le \frac{K}{3!}|0.97-1|^3 < 0.0000024.$$

Since $R_2 < 0$,

$$\begin{aligned}0.7701731 - 0.0000024 < \tan^{-1}(0.97) &< 0.7701731\\ 0.7701707 < \tan^{-1}(0.97) &< 0.7701731.\end{aligned}$$

13. $f(x) = e^x$, $\quad f^{(k)}(x) = e^x$ for $k = 1, 2, 3\ldots$

$a = 0$: $\quad f(x) \approx 1 + x + \dfrac{x^2}{2}$

$e^{-0.5} \approx 1 - 0.5 + \dfrac{(0.5)^2}{2} = 0.625$

Error $= \dfrac{f'''(c)}{6}(0.5)^3 = \dfrac{e^c}{6}(-0.05)^3$ for some c between -0.5 and 0. Thus

$$|\text{Error}| < \frac{(0.5)^3}{6} < 0.020834,$$

and $-0.020833 < e^{-0.5} - 0.625 < 0$, or $0.604 < e^{-0.5} < 0.625$.

14. Since $f(x) = \sin x$, then $f'(x) = \cos x$, $f''(x) = -\sin x$ and $f'''(x) = -\cos x$. Hence,

$$\begin{aligned}\sin(47^\circ) &= f\left(\frac{\pi}{4} + \frac{\pi}{90}\right)\\ &\approx f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(\frac{\pi}{90}\right) + \frac{1}{2}f''\left(\frac{\pi}{4}\right)\left(\frac{\pi}{90}\right)^2\\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{\pi}{90}\right) - \frac{1}{2\sqrt{2}}\left(\frac{\pi}{90}\right)^2\\ &\approx 0.7313587.\end{aligned}$$

The error is $R_2 = \dfrac{f'''(c)}{3!}\left(\dfrac{\pi}{90}\right)^3$ for some c between 45° and 47°. Observe that $R_2 < 0$. If $45^\circ \le t \le 47^\circ$, then

$$|f'''(t)| \le |-\cos 45^\circ| = \frac{1}{\sqrt{2}} = K.$$

Hence,

$$|R_2| \le \frac{K}{3!}\left(\frac{\pi}{90}\right)^3 < 0.0000051.$$

Since $R_2 < 0$, therefore

$$\begin{aligned}0.7313587 - 0.0000051 < \sin(47^\circ) &< 0.7313587\\ 0.7313536 < \sin(47^\circ) &< 0.7313587.\end{aligned}$$

15.
$$\begin{aligned}f(x) &= \sin x\\ f'(x) &= \cos x\\ f''(x) &= -\sin x\\ f'''(x) &= -\cos x\\ f^{(4)}(x) &= \sin x\end{aligned}$$
$a = 0$; $n = 7$:

$$\begin{aligned}\sin x &= 0 + x - 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0 - \frac{x^7}{7!} + R_7,\\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + R_7(x)\end{aligned}$$

where $R_7(x) = \dfrac{\sin c}{8!}x^8$ for some c between 0 and x.

16. For $f(x) = \cos x$ we have

$$\begin{aligned}f'(x) &= -\sin x & f''(x) &= -\cos x & f'''(x) &= \sin x\\ f^{(4)}(x) &= \cos x & f^{(5)}(x) &= -\sin x & f^{(6)}(x) &= -\cos x.\end{aligned}$$

The Taylor's Formula for f with $a = 0$ and $n = 6$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + R_6(f; 0, x)$$

where the Lagrange remainder R_6 is given by

$$R_6 = R_6(f; 0, x) = \frac{f^{(7)}(c)}{7!}x^7 = \frac{\sin c}{7!}x^7,$$

for some c between 0 and x.

17. $f(x) = \sin x \qquad a = \dfrac{\pi}{4}, \qquad n = 4$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2$$
$$-\frac{1}{\sqrt{2}}\frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{\sqrt{2}}\frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + R_4(x)$$

where $R_4(x) = \dfrac{1}{5!}(\cos c)\left(x - \dfrac{\pi}{4}\right)^5$

for some c between $\dfrac{\pi}{4}$ and x.

18. Given that $f(x) = \dfrac{1}{1-x}$, then

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{2}{(1-x)^3}.$$

In general,

$$f^{(n)}(x) = \frac{n!}{(1-x)^{(n+1)}}.$$

Since $a = 0$, $f^{(n)}(0) = n!$. Hence, for $n = 6$, the Taylor's Formula is

$$\begin{aligned}\frac{1}{1-x} &= f(0) + \sum_{n=1}^{6} \frac{f^{(n)}(0)}{n!}x^n + R_6(f; 0, x)\\ &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + R_6(f; 0, x).\end{aligned}$$

The Langrange remainder is

$$R_6(f; 0, x) = \frac{f^{(7)}(c)}{7!}x^7 = \frac{x^7}{(1-c)^8}$$

for some c between 0 and x.

19.

$$\begin{aligned}f(x) &= \ln x\\ f'(x) &= \frac{1}{x}\\ f''(x) &= -\frac{-1}{x^2}\\ f'''(x) &= \frac{2!}{x^3}\\ f^{(4)}(x) &= \frac{-3!}{x^4}\\ f^{(5)}(x) &= \frac{4!}{x^5}\\ f^{(6)}(x) &= \frac{-5!}{x^6}\\ f^{(7)} &= \frac{6!}{x^7}\end{aligned}$$

$a = 1, \quad n = 6$

$$\begin{aligned}\ln x &= 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3\\ &\quad - \frac{3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 - \frac{5!}{6!}(x-1)^6 + R_6(x)\\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}\\ &\quad + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + R_6(x)\end{aligned}$$

where $R_6(x) = \dfrac{1}{7c^7}(x-1)^7$ for some c between 1 and x.

20. Given that $f(x) = \tan x$, then

$$\begin{aligned}f'(x) &= \sec^2 x\\ f''(x) &= 2\sec^2 x \tan x\\ f^{(3)}(x) &= 6\sec^4 x - 4\sec^2 x\\ f^{(4)}(x) &= 8\tan x(3\sec^4 x - \sec^2 x).\end{aligned}$$

Given that $a = 0$ and $n = 3$, the Taylor's Formula is

$$\begin{aligned}\tan x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + R_3(f; 0, x)\\ &= x + \frac{2}{3!}x^3 + R_3(f; 0, x)\\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5.\end{aligned}$$

The Lagrange remainder is

$$R_3(f; 0, x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\tan c(3\sec^4 X - \sec^2 C)}{3}x^4$$

for some c between 0 and x.

21. $e^{3x} = e^{3(x+1)}e^{-3}$

$$P_3(x) = e^{-3}\left[1 + 3(x+1) + \frac{9}{2}(x+1)^2 + \frac{9}{2}(x+1)^3\right].$$

22. For e^u, $P_4(u) = 1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \dfrac{u^4}{4!}$. Let $u = -x^2$. Then for e^{-x^2}:

$$P_8(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}.$$

23. For $\sin^2 x = \dfrac{1}{2}\Big(1 - \cos(2x)\Big)$ at $x = 0$, we have

$$P_4(x) = \frac{1}{2}\left[1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right)\right] = x^2 - \frac{x^4}{3}.$$

24. $\sin x = \sin\Big(\pi + (x-\pi)\Big) = -\sin(x-\pi)$

$$P_5(x) = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!}$$

25. For $\dfrac{1}{1-u}$ at $u = 0$, $P_3(u) = 1 + u + u^2 + u^3$. Let $u = -2x^2$. Then for $\dfrac{1}{1+2x^2}$ at $x = 0$,

$$P_6(x) = 1 - 2x^2 + 4x^4 - 8x^6.$$

26. $\cos(3x - \pi) = -\cos(3x)$

$$P_8(x) = -1 + \frac{3^2x^2}{2!} - \frac{3^4x^4}{4!} + \frac{3^6x^6}{6!} - \frac{3^8x^8}{8!}.$$

27. Since $x^3 = 0 + 0x + 0x^2 + x^3 + 0x^4 + \cdots$ we have $P_n(x) = 0$ if $0 \le n \le 2$; $P_n(x) = x^3$ if $n \ge 3$

28.

29.
$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$
$$\begin{aligned}P_{2n+1}(x) =& \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}\right)\\ &- \frac{1}{2}\left(1 - x + \frac{x^2}{2!} + \cdots - \frac{x^{2n+1}}{(2n+1)!}\right)\\ =& x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}.\end{aligned}$$

30. For $\ln(1+x)$ at $x = 0$ we have
$$P_{2n+1}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n+1}}{2n+1}.$$
For $\ln(1-x)$ at $x = 0$ we have
$$P_{2n+1}(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^{2n+1}}{2n+1}.$$
For $\tanh^{-1} x = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$,
$$P_{2n+1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1}.$$

31. $f(x) = e^{-x}$
$$f^{(n)}(x) = \begin{cases} e^{-x} & \text{if } n \text{ is even} \\ -e^{-x} & \text{if } n \text{ is odd}\end{cases}$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^5}{n!} + R_n(x)$$
where $R_n(x) = (-1)^{n+1}\dfrac{X^{n+1}}{(n+1)!}$ for some X between 0 and x.
For $x = 1$, we have
$$\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!} + R_n(1)$$
where $R_n(1) = (-1)^{n+1}\dfrac{e^{-X}x^{n+1}}{(n+1)!}$ for some X between -1 and 0.
Therefore, $|R_n(1)| < \dfrac{1}{(n+1)!}$. We want $|R_n(1)| < 0.000005$ for 5 decimal places.
Choose n so that $\dfrac{1}{(n+1)!} < 0.000005$. $n = 8$ will do since $1/9! \approx 0.0000027$.
$$\begin{aligned}\text{Thus } \frac{1}{e} &\approx \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}\\ &\approx 0.36788 \quad \text{(to 5 decimal places).}\end{aligned}$$

32. In Taylor's Formulas for $f(x) = \sin x$ with $a = 0$, only odd powers of x have nonzero coefficients. Accordingly we can take terms up to order x^{2n+1} but use the remainder after the next term $0x^{2n+2}$. The formula is
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + R_{2n+2},$$
where
$$R_{2n+2}(f; 0, x) = (-1)^{n+1}\frac{\cos c}{(2n+3)!}x^{2n+3}$$
for some c between 0 and x.
In order to use the formula to approximate $\sin(1)$ correctly to 5 decimal places, we need $|R_{2n+2}(f; 0, 1)| < 0.000005$. Since $|\cos c| \le 1$, it is sufficient to have $1/(2n+3)! < 0.000005$. $n = 3$ will do since $1/9! \approx 0.000003$. Thus
$$\sin(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \approx 0.84147$$
correct to five decimal places.

33. $f(x) = (x-1)^2$, $f'(x) = 2(x-1)$, $f''(x) = 2$.
$f(x) \approx 1 - 2x + \frac{2}{2}x^2 = 1 - 2x + x^2$
Error = 0
$g(x) = x^3 + 2x^2 + 3x + 4$
Quadratic approx.: $g(x) \approx 4 + 3x + 2x^2$
Error $= x^3$
Since $g'''(c) = 6 = 3!$, error $= \dfrac{g'''(c)}{3!}x^3$
so that constant $\dfrac{1}{3!}$ in the error formula for the quadratic approximation cannot be improved.

34. $1 - x^{n+1} = (1-x)(1 + x + x^2 + x^3 + \cdots + x^n)$. Thus
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}.$$
If $|x| \le K < 1$, then $|1-x| \ge 1 - K > 0$, so
$$\left|\frac{x^{n+1}}{1-x}\right| \le \frac{1}{1-K}|x^{n+1}| = O(x^{n+1})$$
as $x \to 0$. By Theorem 11, the nth-order Maclaurin polynomial for $1/(1-x)$ must be $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.

35. Differentiating
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}$$
with respect to x gives
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \frac{n+1-nx}{(1-x)^2}x^n.$$
Then replacing n with $n+1$ gives
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \frac{n+2-(n+1)x}{(1-x)^2}x^{n+1}.$$

If $|x| \le K < 1$, then $|1-x| \ge 1-K > 0$, and so

$$\left|\frac{n+2-(n+1)x}{(1-x)^2}x^{n+1}\right| \le \frac{n+2}{(1-K)^2}|x^{n+1}| = O(x^{n+1})$$

as $x \to 0$. By Theorem 11XXX, the nth-order Maclaurin polynomial for $1/(1-x)^2$ must be $P_n(x) == 1 + 2x + 3x^2 + \cdots + (n+1)x^n$.

Section 4.11 Roundoff Error, Truncation Error, and Computers (page 284)

1. Since the normal rules of algebra (commutativity, associativity, distributivity, etc.) don't apply to floating-point calculations, we should not expect the plots of mathematically equivalent expressions to be the same in all cases.

2. Since $f(x) - P_4(x) = O(|x|^5)$ as x approaches 0, on this very small interval centred at zero we would expect the graph to be the horizontal line through the origin. Instead, there is a band of points having a peculiar structure. The plot can vary between different implementations of Maple on different operating systems, but some of the four horizontal lines (actually envelop curves) proposed in the exercise seem to provide natural boundaries for most of the points.

3. As noted in section 4.7, a real number x can be represented in binary form as,

$$x = \pm 0.d_1d_2\ldots d_td_{t+1}d_{t+2}\ldots \times 2^p$$

where each of the base two digits d_i is either 0 or 1, but $d_1 = 1$, and p is the appropriate power of 2.

Consider the floating point number, y (having the same sign as x) given by

$$y = \pm 0.d_1d_2\ldots d_t \times 10^p,$$

which has only t significant binary digits. The distance between x and y thus satisfies

$$|x-y| = 0.d_{t+1}d_{t+2}\cdots \times 2^{p-t}.$$

Since d_{t+1} may or may not be 1, this distance is less than or equal to 2^{p-t}. Thus $F(x)$, the *nearest* floating point number to x, can be no further than half that distance away, or

$$|x - F(x)| \le \frac{1}{2}2^{p-t} = 2^{p-t-1}.$$

It follows that

$$|x - F(x)| \le 2^{p-t-1} \le 2^{-t}|x|$$

because $|x| \ge 2^{p-1}$. Clearly 2^{-t} is the smallest value when added to 1 that will not be discarded thus

$$|F(x) - x| \le \epsilon|x|.$$

4.

a) The result of question 3 suggests that computer produces a number $C(1+\alpha)$ for one expression and $C(1+\beta)$ for the other. α and β will both be discarded positive numbers that are both less than ϵ by assumption. Thus for each expression, individually the computer returns C, however the difference between these two expressions is $(\alpha-\beta)C$, where $|(\alpha-\beta)| < \epsilon$. Thus the computer returns a value for the difference which is less than ϵC but not necessarily equal to 0.

b) Yes. In certain cases, internal errors may grow sufficiently that the computer returns a value different from C, meaning that either or both of α and β may not be entirely discarded. The algorithm evaluating such expression may be in need of improvement in such a case.

c) Yes. In certain cases internal errors can be negligible or zero, or they can cancel one another. Consider for example the expression. 1 - $\sqrt{1^2}$, which will not challenge machine precision at all.

Review Exercises 4 (page 285)

1. Since $dr/dt = 2r/100$ and $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = \frac{4\pi}{3}3r^2\frac{dr}{dt} = 3V\frac{2}{100} = \frac{6V}{100}.$$

Hence The volume is increasing at 6%/min.

2. a) Since F must be continuous at $r = R$, we have

$$\frac{mgR^2}{R^2} = mkR, \quad \text{or} \quad k = \frac{g}{R}.$$

b) The rate of change of F as r decreases from R is

$$\left(-\frac{d}{dr}(mkr)\right)\bigg|_{r=R} = -mk = -\frac{mg}{R}.$$

The rate of change of F as r increases from R is

$$\left(-\frac{d}{dr}\frac{mgR^2}{r^2}\right)\bigg|_{r=R} = -\frac{2mgR^2}{R^3} = -2\frac{mg}{R}.$$

Thus F decreases as r increases from R at twice the rate at which it decreases as r decreases from R.