Section 2.10 Antiderivatives and Initial-Value Problems (page 154)

1. $\int 5\,dx = 5x + C$

2. $\int x^2\,dx = \frac{1}{3}x^3 + C$

3. $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

4. $\int x^{12}\,dx = \frac{1}{13}x^{13} + C$

5. $\int x^3\,dx = \frac{1}{4}x^4 + C$

6. $\int (x + \cos x)\,dx = \frac{x^2}{2} + \sin x + C$

7. $\int \tan x \cos x\,dx = \int \sin x\,dx = -\cos x + C$

8. $\int \frac{1 + \cos^3 x}{\cos^2 x}\,dx = \int (\sec^2 x + \cos x)\,dx = \tan x + \sin x + C$

9. $\int (a^2 - x^2)\,dx = a^2 x - \frac{1}{3}x^3 + C$

10. $\int (A + Bx + Cx^2)\,dx = Ax + \frac{B}{2}x^2 + \frac{C}{3}x^3 + K$

11. $\int (2x^{1/2} + 3x^{1/3}\,dx = \frac{4}{3}x^{3/2} + \frac{9}{4}x^{4/3} + C$

12. $$\int \frac{6(x-1)}{x^{4/3}}\,dx = \int (6x^{-1/3} - 6x^{-4/3})\,dx$$
$$= 9x^{2/3} + 18x^{-1/3} + C$$

13. $\int \left(\frac{x^3}{3} - \frac{x^2}{2} + x - 1\right)dx = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + C$

14. $$105 \int (1 + t^2 + t^4 + t^6)\,dt$$
$$= 105(t + \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + \tfrac{1}{7}t^7) + C$$
$$= 105t + 35t^3 + 21t^5 + 15t^7 + C$$

15. $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$

16. $\int \sin\left(\frac{x}{2}\right)dx = -2\cos\left(\frac{x}{2}\right) + C$

17. $\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x} + C$

18. $\int \sec(1-x)\tan(1-x)\,dx = -\sec(1-x) + C$

19. $\int \sqrt{2x+3}\,dx = \frac{1}{3}(2x+3)^{3/2} + C$

20. Since $\frac{d}{dx}\sqrt{x+1} = \frac{1}{2\sqrt{x+1}}$, therefore
$$\int \frac{4}{\sqrt{x+1}}\,dx = 8\sqrt{x+1} + C.$$

21. $\int 2x\sin(x^2)\,dx = -\cos(x^2) + C$

22. Since $\frac{d}{dx}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$, therefore
$$\int \frac{2x}{\sqrt{x^2+1}}\,dx = 2\sqrt{x^2+1} + C.$$

23. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

24. $\int \sin x \cos x\,dx = \int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x) + C$

25. $\int \cos^2 x\,dx = \int \frac{1 + \cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$

26. $\int \sin^2 x\,dx = \int \frac{1 - \cos(2x)}{2}\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$

27. $$\begin{cases} y' = x - 2 & \Rightarrow y = \frac{1}{2}x^2 - 2x + C \\ y(0) = 3 & \Rightarrow 3 = 0 + C \text{ therefore } C = 3 \end{cases}$$
Thus $y = \frac{1}{2}x^2 - 2x + 3$ for all x.

28. Given that
$$\begin{cases} y' = x^{-2} - x^{-3} \\ y(-1) = 0, \end{cases}$$
then $y = \int (x^{-2} - x^{-3})\,dx = -x^{-1} + \frac{1}{2}x^{-2} + C$ and $0 = y(-1) = -(-1)^{-1} + \frac{1}{2}(-1)^{-2} + C$ so $C = -\frac{3}{2}$. Hence, $y(x) = -\frac{1}{x} + \frac{1}{2x^2} - \frac{3}{2}$ which is valid on the interval $(-\infty, 0)$.

29. $$\begin{cases} y' = 3\sqrt{x} & \Rightarrow \quad y = 2x^{3/2} + C \\ y(4) = 1 & \Rightarrow \quad 1 = 16 + C \text{ so } C = -15 \end{cases}$$
Thus $y = 2x^{3/2} - 15$ for $x > 0$.

30. Given that
$$\begin{cases} y' = x^{1/3} \\ y(0) = 5, \end{cases}$$
then $y = \int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + C$ and $5 = y(0) = C$. Hence, $y(x) = \frac{3}{4}x^{4/3} + 5$ which is valid on the whole real line.

31. Since $y' = Ax^2 + Bx + C$ we have $y = \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx + D$. Since $y(1) = 1$, therefore $1 = y(1) = \frac{A}{3} + \frac{B}{2} + C + D$. Thus $D = 1 - \frac{A}{3} - \frac{B}{2} - C$, and
$$y = \frac{A}{3}(x^3 - 1) + \frac{B}{2}(x^2 - 1) + C(x - 1) + 1 \text{ for all } x$$

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2. $x = 4 + 5t - t^2$, $v = 5 - 2t$, $a = -2$.

a) The point is moving to the right if $v > 0$, i.e., when $t < \frac{5}{2}$.

b) The point is moving to the left if $v < 0$, i.e., when $t > \frac{5}{2}$.

c) The point is accelerating to the right if $a > 0$, but $a = -2$ at all t; hence, the point never accelerates to the right.

d) The point is accelerating to the left if $a < 0$, i.e., for all t.

e) The particle is speeding up if v and a have the same sign, i.e., for $t > \frac{5}{2}$.

f) The particle is slowing down if v and a have opposite sign, i.e., for $t < \frac{5}{2}$.

g) Since $a = -2$ at all t, $a = -2$ at $t = \frac{5}{2}$ when $v = 0$.

h) The average velocity over $[0, 4]$ is $\dfrac{x(4) - x(0)}{4} = \dfrac{8-4}{4} = 1$.

3. $x = t^3 - 4t + 1$, $v = \dfrac{dx}{dt} = 3t^2 - 4$, $a = \dfrac{dv}{dt} = 6t$

a) particle moving: to the right for $t < -2/\sqrt{3}$ or $t > 2/\sqrt{3}$,

b) to the left for $-2/\sqrt{3} < t < 2/\sqrt{3}$

c) particle is accelerating: to the right for $t > 0$

d) to the left for $t < 0$

e) particle is speeding up for $t > 2/\sqrt{3}$ or for $-2/\sqrt{3} < t < 0$

f) particle is slowing down for $t < -2/\sqrt{3}$ or for $0 < t < 2/\sqrt{3}$

g) velocity is zero at $t = \pm 2/\sqrt{3}$. Acceleration at these times is $\pm 12/\sqrt{3}$.

h) average velocity on $[0, 4]$ is $\dfrac{4^3 - 4 \times 4 + 1 - 1}{4 - 0} = 12$

4. $x = \dfrac{t}{t^2+1}$, $\quad v = \dfrac{(t^2+1)(1) - (t)(2t)}{(t^2+1)^2} = \dfrac{1-t^2}{(t^2+1)^2}$,

$$a = \frac{(t^2+1)^2(-2t) - (1-t^2)(2)(t^2+1)(2t)}{(t^2+1)^4} = \frac{2t(t^2-3)}{(t^2+1)^3}.$$

a) The point is moving to the right if $v > 0$, i.e., when $1 - t^2 > 0$, or $-1 < t < 1$.

b) The point is moving to the left if $v < 0$, i.e., when $t < -1$ or $t > 1$.

c) The point is accelerating to the right if $a > 0$, i.e., when $2t(t^2 - 3) > 0$, that is, when $t > \sqrt{3}$ or $-\sqrt{3} < t < 0$.

d) The point is accelerating to the left if $a < 0$, i.e., for $t < -\sqrt{3}$ or $0 < t < \sqrt{3}$.

e) The particle is speeding up if v and a have the same sign, i.e., for $t < -\sqrt{3}$, or $-1 < t < 0$ or $1 < t < \sqrt{3}$.

f) The particle is slowing down if v and a have opposite sign, i.e., for $-\sqrt{3} < t < -1$, or $0 < t < 1$ or $t > \sqrt{3}$.

g) $v = 0$ at $t = \pm 1$. At $t = -1$, $a = \dfrac{-2(-2)}{(2)^3} = \dfrac{1}{2}$.
At $t = 1$, $a = \dfrac{2(-2)}{(2)^3} = -\dfrac{1}{2}$.

h) The average velocity over $[0, 4]$ is $\dfrac{x(4) - x(0)}{4} = \dfrac{\frac{4}{17} - 0}{4} = \dfrac{1}{17}$.

5. $y = 9.8t - 4.9t^2$ metres (t in seconds)
velocity $v = \dfrac{dy}{dt} = 9.8 - 9.8t$
acceleration $a = \dfrac{dv}{dt} = -9.8$
The acceleration is 9.8 m/s^2 downward at all times.
Ball is at maximum height when $v = 0$, i.e., at $t = 1$.
Thus maximum height is $y\Big|_{t=1} = 9.8 - 4.9 = 4.9$ metres.
Ball strikes the ground when $y = 0$, ($t > 0$), i.e., $0 = t(9.8 - 4.9t)$ so $t = 2$.
Velocity at $t = 2$ is $9.8 - 9.8(2) = -9.8$ m/s.
Ball strikes the ground travelling at 9.8 m/s (downward).

6. Given that $y = 100 - 2t - 4.9t^2$, the time t at which the ball reaches the ground is the positive root of the equation $y = 0$, i.e., $100 - 2t - 4.9t^2 = 0$, namely,

$$t = \frac{-2 + \sqrt{4 + 4(4.9)(100)}}{9.8} \approx 4.318 \text{ s}.$$

The average velocity of the ball is $\dfrac{-100}{4.318} = -23.16$ m/s.
Since $-23.159 = v = -2 - 9.8t$, then $t \simeq 2.159$ s.

7. $D = t^2$, D in metres, t in seconds
velocity $v = \dfrac{dD}{dt} = 2t$
Aircraft becomes airborne if
$v = 200$ km/h $= \dfrac{200,000}{3600} = \dfrac{500}{9}$ m/s.
Time for aircraft to become airborne is $t = \dfrac{250}{9}$ s, that is, about 27.8 s.
Distance travelled during takeoff run is $t^2 \approx 771.6$ metres.

8. Let $y(t)$ be the height of the projectile t seconds after it is fired upward from ground level with initial speed v_0. Then

$$y''(t) = -9.8, \quad y'(0) = v_0, \quad y(0) = 0.$$

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10. csch^{-1} has domain and range consisting of all real numbers x except $x = 0$. We have

$$\frac{d}{dx}\operatorname{csch}^{-1} x = \frac{d}{dx}\sinh^{-1}\frac{1}{x}$$
$$= \frac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}}\left(\frac{-1}{x^2}\right) = \frac{-1}{|x|\sqrt{x^2+1}}.$$

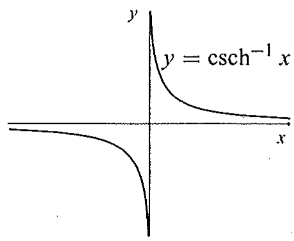

Fig. 3.6.10

11. $f_{A,B}(x) = Ae^{kx} + Be^{-kx}$
$f'_{A,B}(x) = kAe^{kx} - kBe^{-kx}$
$f''_{A,B}(x) = k^2Ae^{kx} + k^2Be^{-kx}$
Thus $f''_{A,B} - k^2 f_{A,B} = 0$

$$g_{C,D}(x) = C\cosh kx + D\sinh kx$$
$$g'_{C,D}(x) = kC\cosh kx + kD\sinh kx$$
$$g''_{C,D}(x) = k^2C\cosh kx + k^2D\sinh kx$$

Thus $g''_{C,D} - k^2 g_{C,D} = 0$
$\cosh kx + \sinh kx = e^{kx}$
$\cosh kx - \sinh kx = e^{-kx}$
Thus $f_{A,B}(x) = (A+B)\cosh kx + (A-B)\sinh kx$, that is,
$f_{A,B}(x) = g_{A+B,A-B}(x)$, and
$g_{C,D}(x) = \frac{C}{2}(e^{kx} + e^{-kx}) + \frac{D}{2}(e^{kx} - e^{-kx})$,
that is $g_{C,D}(x) = f_{(C+D)/2,(C-D)/2}(x)$.

12. Since

$$h_{L,M}(x) = L\cosh k(x-a) + M\sinh k(x-a)$$
$$h''_{L,M}(x) = Lk^2\cosh k(x-a) + Mk^2\sinh k(x-a)$$
$$= k^2 h_{L,M}(x)$$

hence, $h_{L,M}(x)$ is a solution of $y'' - k^2y = 0$ and

$$h_{L,M}(x)$$
$$= \frac{L}{2}\left(e^{kx-ka} + e^{-kx+ka}\right) + \frac{M}{2}\left(e^{kx-ka} - e^{-kx+ka}\right)$$
$$= \left(\frac{L}{2}e^{-ka} + \frac{M}{2}e^{-ka}\right)e^{kx} + \left(\frac{L}{2}e^{ka} - \frac{M}{2}e^{ka}\right)e^{-kx}$$
$$= Ae^{kx} + Be^{-kx} = f_{A,B}(x)$$

where $A = \frac{1}{2}e^{-ka}(L+M)$ and $B = \frac{1}{2}e^{ka}(L-M)$.

13. $y'' - k^2y = 0 \Rightarrow y = h_{L,M}(x)$
$= L\cosh k(x-a) + M\sinh k(x-a)$
$y(a) = y_0 \Rightarrow y_0 = L + 0 \Rightarrow L = y_0$,
$y'(a) = v_0 \Rightarrow v_0 = 0 + Mk \Rightarrow M = \dfrac{v_0}{k}$
Therefore $y = h_{y_0,v_0/k}(x)$
$= y_0\cosh k(x-a) + (v_0/k)\sinh k(x-a)$.

Section 3.7 Second-Order Linear DEs with Constant Coefficients (page 210)

1. $y'' + 7y' + 10y = 0$
auxiliary eqn $r^2 + 7r + 10 = 0$
$(r+5)(r+2) = 0 \quad\Rightarrow\quad r = -5, -2$
$y = Ae^{-5t} + Be^{-2t}$

2. $y'' - 2y' - 3y = 0$
auxiliary eqn $r^2 - 2r - 3 = 0 \quad\Rightarrow\quad r = -1,\ r = 3$
$y = Ae^{-t} + Be^{3t}$

3. $y'' + 2y' = 0$
auxiliary eqn $r^2 + 2r = 0 \quad\Rightarrow\quad r = 0,\ -2$
$y = A + Be^{-2t}$

4. $4y'' - 4y' - 3y = 0$
$4r^2 - 4r - 3 = 0 \Rightarrow (2r+1)(2r-3) = 0$
Thus, $r_1 = -\frac{1}{2}$, $r_2 = \frac{3}{2}$, and $y = Ae^{-(1/2)t} + Be^{(3/2)t}$.

5. $y'' + 8y' + 16y = 0$
auxiliary eqn $r^2 + 8r + 16 = 0 \quad\Rightarrow\quad r = -4,\ -4$
$y = Ae^{-4t} + Bte^{-4t}$

6. $y'' - 2y' + y = 0$
$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$
Thus, $r = 1,\ 1$, and $y = Ae^t + Bte^t$.

7. $y'' - 6y' + 10y = 0$
auxiliary eqn $r^2 - 6r + 10 = 0 \quad\Rightarrow\quad r = 3 \pm i$
$y = Ae^{3t}\cos t + Be^{3t}\sin t$

8. $9y'' + 6y' + y = 0$
$9r^2 + 6r + 1 = 0 \Rightarrow (3r+1)^2 = 0$
Thus, $r = -\frac{1}{3},\ -\frac{1}{3}$, and $y = Ae^{-(1/3)t} + Bte^{-(1/3)t}$.

9. $y'' + 2y' + 5y = 0$
auxiliary eqn $r^2 + 2r + 5 = 0 \quad\Rightarrow\quad r = -1 \pm 2i$
$y = Ae^{-t}\cos 2t + Be^{-t}\sin 2t$

10. For $y'' - 4y' + 5y = 0$ the auxiliary equation is $r^2 - 4r + 5 = 0$, which has roots $r = 2 \pm i$. Thus, the general solution of the DE is $y = Ae^{2t}\cos t + Be^{2t}\sin t$.

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or, more simply, $u'' - (r_2 - r_1)u' = 0$. Putting $v = u'$ reduces this equation to first order:

$$v' = (r_2 - r_1)v,$$

which has general solution $v = Ce^{(r_2-r_1)t}$. Hence

$$u = \int Ce^{(r_2-r_1)t}\,dt = Be^{(r_2-r_1)t} + A,$$

and $y = e^{r_1 t}u = Ae^{r_1 t} + Be^{r_2 t}$.

19. If $y = A\cos\omega t + B\sin\omega t$ then

$$\begin{aligned} y'' + \omega^2 y &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t \\ &\quad + \omega^2(A\cos\omega t + B\sin\omega t) = 0 \end{aligned}$$

for all t. So y is a solution of (†).

20. If $f(t)$ is any solution of (†) then $f''(t) = -\omega^2 f(t)$ for all t. Thus,

$$\begin{aligned} &\frac{d}{dt}\left[\omega^2\big(f(t)\big)^2 + \big(f'(t)\big)^2\right] \\ &= 2\omega^2 f(t)f'(t) + 2f'(t)f''(t) \\ &= 2\omega^2 f(t)f'(t) - 2\omega^2 f(t)f'(t) = 0 \end{aligned}$$

for all t. Thus, $\omega^2\big(f(t)\big)^2 + \big(f'(t)\big)^2$ is constant. (This can be interpreted as a conservation of energy statement.)

21. If $g(t)$ satisfies (†) and also $g(0) = g'(0) = 0$, then by Exercise 20,

$$\begin{aligned} &\omega^2\big(g(t)\big)^2 + \big(g'(t)\big)^2 \\ &= \omega^2\big(g(0)\big)^2 + \big(g'(0)\big)^2 = 0. \end{aligned}$$

Since a sum of squares cannot vanish unless each term vanishes, $g(t) = 0$ for all t.

22. If $f(t)$ is any solution of (†), let $g(t) = f(t) - A\cos\omega t - B\sin\omega t$ where $A = f(0)$ and $B\omega = f'(0)$. Then g is also solution of (†). Also $g(0) = f(0) - A = 0$ and $g'(0) = f'(0) - B\omega = 0$. Thus, $g(t) = 0$ for all t by Exercise 24, and therefore $f(x) = A\cos\omega t + B\sin\omega t$. Thus, it is proved that every solution of (†) is of this form.

23. We are given that $k = -\dfrac{b}{2a}$ and $\omega^2 = \dfrac{4ac - b^2}{4a^2}$ which is positive for Case III. If $y = e^{kt}u$, then

$$\begin{aligned} y' &= e^{kt}\big(u' + ku\big) \\ y'' &= e^{kt}\big(u'' + 2ku' + k^2u\big). \end{aligned}$$

Substituting into $ay'' + by' + cy = 0$ leads to

$$\begin{aligned} 0 &= e^{kt}\Big(au'' + (2ka + b)u' + (ak^2 + bk + c)u\Big) \\ &= e^{kt}\Big(au'' + 0 + ((b^2/(4a) - (b^2/(2a) + c)u\Big) \\ &= a\,e^{kt}\Big(u'' + \omega^2 u\Big). \end{aligned}$$

Thus u satisfies $u'' + \omega^2 u = 0$, which has general solution

$$u = A\cos(\omega t) + B\sin(\omega t)$$

by the previous problem. Therefore $ay'' + by' + cy = 0$ has general solution

$$y = Ae^{kt}\cos(\omega t) + Be^{kt}\sin(\omega t).$$

24. Because $y'' + 4y = 0$, therefore $y = A\cos 2t + B\sin 2t$. Now

$$\begin{aligned} y(0) = 2 &\Rightarrow A = 2, \\ y'(0) = -5 &\Rightarrow B = -\tfrac{5}{2}. \end{aligned}$$

Thus, $y = 2\cos 2t - \frac{5}{2}\sin 2t$.
circular frequency $= \omega = 2$, frequency $= \dfrac{\omega}{2\pi} = \dfrac{1}{\pi} \approx 0.318$
period $= \dfrac{2\pi}{\omega} = \pi \approx 3.14$
amplitude $= \sqrt{(2)^2 + (-\frac{5}{2})^2} \simeq 3.20$

25. $\begin{cases} y'' + 100y = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$
$y = A\cos(10t) + B\sin(10t)$
$A = y(0) = 0, \quad 10B = y'(0) = 3$
$y = \dfrac{3}{10}\sin(10t)$

26. $y = \mathcal{A}\cos\big(\omega(t - c)\big) + \mathcal{B}\sin\big(\omega(t - c)\big)$
(easy to calculate $y'' + \omega^2 y = 0$)

$$\begin{aligned} y &= \mathcal{A}\Big(\cos(\omega t)\cos(\omega c) + \sin(\omega t)\sin(\omega c)\Big) \\ &\quad + \mathcal{B}\Big(\sin(\omega t)\cos(\omega c) - \cos(\omega t)\sin(\omega c)\Big) \\ &= \Big(\mathcal{A}\cos(\omega c) - \mathcal{B}\sin(\omega c)\Big)\cos\omega t \\ &\quad + \Big(\mathcal{A}\sin(\omega c) + \mathcal{B}\cos(\omega c)\Big)\sin\omega t \\ &= A\cos\omega t + B\sin\omega t \end{aligned}$$

where $A = \mathcal{A}\cos(\omega c) - \mathcal{B}\sin(\omega c)$ and $B = \mathcal{A}\sin(\omega c) + \mathcal{B}\cos(\omega c)$

27. For $y'' + y = 0$, we have $y = A\sin t + B\cos t$. Since,

$$\begin{aligned} y(2) &= 3 = A\sin 2 + B\cos 2 \\ y'(2) &= -4 = A\cos 2 - B\sin 2, \end{aligned}$$

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