Therefore $F(x) = \int x^2\,dx = \frac{1}{3}x^3 + C$. Since $F(0) = C = 0$, therefore $F(x) = \frac{1}{3}x^3$. For $x = 2$, the area of the region is $F(2) = \frac{8}{3}$ square units.

75. a) The shaded area A in part (i) of the figure is less than the area of the rectangle (actually a square) with base from $t = 1$ to $t = 2$ and height $1/1 = 1$. Since $\ln 2 = A < 1$, we have $2 < e^1 = e$; i.e., $e > 2$.

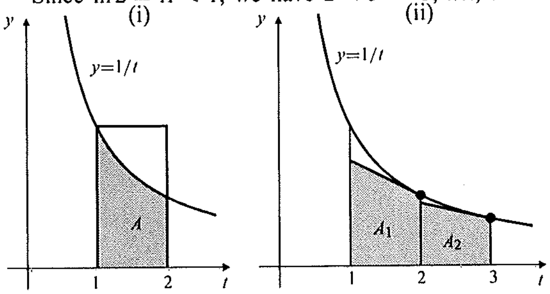

Fig. 3.3.75

b) If $f(t) = 1/t$, then $f'(t) = -1/t^2$ and $f''(t) = 2/t^3 > 0$ for $t > 0$. Thus $f'(t)$ is an increasing function of t for $t > 0$, and so the graph of $f(t)$ bends upward away from any of its tangent lines. (This kind of argument will be explored further in Chapter 5.)

c) The tangent to $y = 1/t$ at $t = 2$ has slope $-1/4$. Its equation is

$$y = \frac{1}{2} - \frac{1}{4}(x-2) \quad \text{or } y = 1 - \frac{x}{4}.$$

The tangent to $y = 1/t$ at $t = 3$ has slope $-1/9$. Its equation is

$$y = \frac{1}{3} - \frac{1}{9}(x-3) \quad \text{or } y = \frac{2}{3} - \frac{x}{9}.$$

d) The trapezoid bounded by $x = 1$, $x = 2$, $y = 0$, and $y = 1 - (x/4)$ has area

$$A_1 = \frac{1}{2}\left(\frac{3}{4} + \frac{1}{2}\right) = \frac{5}{8}.$$

The trapezoid bounded by $x = 2$, $x = 3$, $y = 0$, and $y = (2/3) - (x/9)$ has area

$$A_2 = \frac{1}{2}\left(\frac{4}{9} + \frac{1}{3}\right) = \frac{7}{18}.$$

e) $\ln 3 > A_1 + A_2 = \frac{5}{8} + \frac{7}{18} = \frac{73}{72} > 1$.
Thus $3 > e^1 = e$. Combining this with the result of (a) we conclude that $2 < e < 3$.

Section 3.4 Growth and Decay (page 189)

1. $\lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x} = 0$ (exponential wins)

2. $\lim_{x\to\infty} x^{-3} e^x = \lim_{x\to\infty} \frac{e^x}{x^3} = \infty$

3. $\lim_{x\to\infty} \frac{2e^x - 3}{e^x + 5} = \lim_{x\to\infty} \frac{2 - 3e^{-x}}{1 + 5e^{-x}} = \frac{2-0}{1+0} = 2$

4. $\lim_{x\to\infty} \frac{x - 2e^{-x}}{x + 3e^{-x}} = \lim_{x\to\infty} \frac{1 - 2/(xe^x)}{1 + 3/(xe^x)} = \frac{1-0}{1+0} = 1$

5. $\lim_{x\to 0+} x \ln x = 0$ (power wins)

6. $\lim_{x\to 0+} \frac{\ln x}{x} = -\infty$

7. $\lim_{x\to 0} x(\ln|x|)^2 = 0$

8. $\lim_{x\to\infty} \frac{(\ln x)^3}{\sqrt{x}} = 0$ (power wins)

9. Let $N(t)$ be the number of bacteria present after t hours.
Then $N(0) = 100$, $N(1) = 200$.
Since $\frac{dN}{dt} = kN$ we have $N(t) = N(0)e^{kt} = 100e^{kt}$.
Thus $200 = 100e^k$ and $k = \ln 2$.
Finally, $N\left(\frac{5}{2}\right) = 100e^{(5/2)\ln 2} \approx 565.685$.
There will be approximately 566 bacteria present after another $1\frac{1}{2}$ hours.

10. Let $y(t)$ be the number of kg undissolved after t hours. Thus, $y(0) = 50$ and $y(5) = 20$. Since $y'(t) = ky(t)$, therefore $y(t) = y(0)e^{kt} = 50e^{kt}$. Then

$$20 = y(5) = 50e^{5k} \Rightarrow k = \frac{1}{5}\ln\frac{2}{5}.$$

If 90% of the sugar is dissolved at time T then $5 = y(T) = 50e^{kT}$, so

$$T = \frac{1}{k}\ln\frac{1}{10} = \frac{5\ln(0.1)}{\ln(0.4)} \approx 12.56.$$

Hence, 90% of the sugar will dissolved in about 12.56 hours.

11. Let $P(t)$ be the percentage undecayed after t years.
Thus $P(0) = 100$, $P(15) = 70$.
Since $\frac{dP}{dt} = kP$, we have $P(t) = P(0)e^{kt} = 100e^{kt}$.
Thus $70 = P(15) = 100e^{15k}$ so $k = \frac{1}{15}\ln(0.7)$.
The half-life T satisfies if $50 = P(T) = 100e^{kT}$, so
$T = \frac{1}{k}\ln(0.5) = \frac{15\ln(0.5)}{\ln(0.7)} \approx 29.15$.
The half-life is about 29.15 years.

12. Let $P(t)$ be the percentage remaining after t years. Thus $P'(t) = kP(t)$ and $P(t) = P(0)e^{kt} = 100e^{kt}$. Then,

$$50 = P(1690) = 100e^{1690k} \Rightarrow k = \frac{1}{1690}\ln\frac{1}{2} \approx 0.0004101.$$

a) $P(100) = 100e^{100k} \approx 95.98$, i.e., about 95.98% remains after 100 years.

b) $P(1000) = 100e^{1000k} \approx 66.36$, i.e., about 66.36% remains after 1000 years.

13. Let $P(t)$ be the percentage of the initial amount remaining after t years.
Then $P(t) = 100e^{kt}$ and $99.57 = P(1) = 100e^k$.
Thus $k = \ln(0.9957)$.
The half-life T satisfies $50 = P(T) = 100e^{kT}$,
so $T = \frac{1}{k}\ln(0.5) = \frac{\ln(0.5)}{\ln(0.995)} \approx 160.85$.
The half-life is about 160.85 years.

14. Let $N(t)$ be the number of bacteria in the culture t days after the culture was set up. Thus $N(3) = 3N(0)$ and $N(7) = 10 \times 10^6$. Since $N(t) = N(0)e^{kt}$, we have

$$3N(0) = N(3) = N(0)e^{3k} \Rightarrow k = \tfrac{1}{3}\ln 3.$$
$$10^7 = N(7) = N(0)e^{7k} \Rightarrow N(0) = 10^7 e^{-(7/3)\ln 3} \approx 770400.$$

There were approximately 770,000 bacteria in the culture initially. (Note that we are approximating a discrete quantity (number of bacteria) by a continuous quantity $N(t)$ in this exercise.)

15. Let $W(t)$ be the weight t days after birth.
Thus $W(0) = 4000$ and $W(t) = 4000e^{kt}$.
Also $4400 = W(14) = 4000e^{14k}$, is $k = \frac{1}{14}\ln(1.1)$.
Five days after birth, the baby weighs
$W(5) = 4000e^{(5/14)\ln(1.1)} \approx 4138.50 \approx 4139$ grams.

16. Since

$$I'(t) = kI(t) \Rightarrow I(t) = I(0)e^{kt} = 40e^{kt},$$
$$15 = I(0.01) = 40e^{0.01k} \Rightarrow k = \frac{1}{0.01}\ln\frac{15}{40} = 100\ln\frac{3}{8},$$

thus,

$$I(t) = 40\exp\left(100t\ln\frac{3}{8}\right) = 40\left(\frac{3}{8}\right)^{100t}.$$

17. $\$P$ invested at 4% compounded continuously grows to $\$P(e^{0.04})^7 = \$Pe^{0.28}$ in 7 years. This will be \$10,000 if $\$P = \$10,000e^{-0.28} = \$7,557.84$.

18. Let $y(t)$ be the value of the investment after t years. Thus $y(0) = 1000$ and $y(5) = 1500$. Since $y(t) = 1000e^{kt}$ and $1500 = y(5) = 1000e^{5k}$, therefore, $k = \frac{1}{5}\ln\frac{3}{2}$.

a) Let t be the time such that $y(t) = 2000$, i.e.,

$$1000e^{kt} = 2000$$
$$\Rightarrow \quad t = \frac{1}{k}\ln 2 = \frac{5\ln 2}{\ln(\frac{3}{2})} = 8.55.$$

Hence, the doubling time for the investment is about 8.55 years.

b) Let r% be the effective annual rate of interest; then

$$1000(1 + \frac{r}{100}) = y(1) = 1000e^k$$
$$\Rightarrow r = 100(e^k - 1) = 100[\exp(\tfrac{1}{5}\ln\tfrac{3}{2}) - 1]$$
$$= 8.447.$$

The effective annual rate of interest is about 8.45%.

19. Let the purchasing power of the dollar be $P(t)$ cents after t years.
Then $P(0) = 100$ and $P(t) = 100e^{kt}$.
Now $91 = P(1) = 100e^k$ so $k = \ln(0.91)$.
If $25 = P(t) = 100^{kt}$ then
$t = \frac{1}{k}\ln(0.25) = \frac{\ln(0.25)}{\ln(0.91)} \approx 14.7$.
The purchasing power will decrease to \$0.25 in about 14.7 years.

20. Let i% be the effective rate, then an original investment of $\$A$ will grow to $\$A\left(1 + \frac{i}{100}\right)$ in one year. Let r% be the nominal rate per annum compounded n times per year, then an original investment of $\$A$ will grow to

$$\$A\left(1 + \frac{r}{100n}\right)^n$$

in one year, if compounding is performed n times per year. For $i = 9.5$ and $n = 12$, we have

$$\$A\left(1 + \frac{9.5}{100}\right) = \$A\left(1 + \frac{r}{1200}\right)^{12}$$
$$\Rightarrow r = 1200\left(\sqrt[12]{1.095} - 1\right) = 9.1098.$$

The nominal rate of interest is about 9.1098%.

21. Let $x(t)$ be the number of rabbits on the island t years after they were introduced. Thus $x(0) = 1,000$, $x(3) = 3,500$, and $x(7) = 3,000$. For $t < 5$ we have $dx/dt = k_1 x$, so

$$x(t) = x(0)e^{k_1 t} = 1,000e^{k_1 t}$$
$$x(2) = 1,000e^{2k_1} = 3,500 \quad \Longrightarrow \quad e^{2k_1} = 3.5$$
$$x(5) = 1,000e^{5k_1} = 1,000\left(e^{2k_1}\right)^{5/2} = 1,000(3.5)^{5/2}$$
$$\approx 22,918.$$

For $t > 5$ we have $dx/dt = k_2 x$, so that

$$\begin{aligned}
x(t) &= x(5)e^{k_2(t-5)}\\
x(7) &= x(5)e^{2k_2} = 3{,}000 \quad\Longrightarrow\quad e^{2k_2} \approx \frac{3{,}000}{22{,}918}\\
x(10) &= x(5)3^{5k_2} = x(5)\left(e^{2k_2}\right)^{5/2} \approx 22{,}918\left(\frac{3{,}000}{22{,}918}\right)^{5/2}\\
&\approx 142.
\end{aligned}$$

so there are approximately 142 rabbits left after 10 years.

22. Let $N(t)$ be the number of rats on the island t months after the initial population was released and before the first cull. Thus $N(0) = R$ and $N(3) = 2R$. Since $N(t) = Re^{kt}$, we have $e^{3k} = 2$, so $e^k = 2^{1/3}$. Hence $N(5) = Re^{5k} = 2^{5/3}R$. After the first 1,000 rats are killed the number remaining is $2^{5/3}R - 1{,}000$. If this number is less than R, the number at the end of succeeding 5-year periods will decline. The minimum value of R for which this won't happen must satisfy $2^{5/3}R - 1{,}000 = R$, that is, $R = 1{,}000/(2^{5/3} - 1) \approx 459.8$. Thus $R = 460$ rats should be brought to the island initially.

23. $f'(x) = a + bf(x)$.

a) If $u(x) = a + bf(x)$, then $u'(x) = bf'(x) = b[a + bf(x)] = bu(x)$. This equation for u is the equation of exponential growth/decay. Thus

$$\begin{aligned}
u(x) &= C_1 e^{bx},\\
f(x) &= \frac{1}{b}\left(C_1 e^{bx} - a\right) = Ce^{bx} - \frac{a}{b}.
\end{aligned}$$

b) If $\dfrac{dy}{dx} = a + by$ and $y(0) = y_0$, then, from part (a),

$$y = Ce^{bx} - \frac{a}{b}, \qquad y_0 = Ce^0 - \frac{a}{b}.$$

Thus $C = y_0 + (a/b)$, and

$$y = \left(y_0 + \frac{a}{b}\right)e^{bx} - \frac{a}{b}.$$

24. a) The concentration $x(t)$ satisfies $\dfrac{dx}{dt} = a - bx(t)$. This says that $x(t)$ is increasing if it is less than a/b and decreasing if it is greater than a/b. Thus, the limiting concentration is a/b.

b) The differential equation for $x(t)$ resembles that of Exercise 21(b), except that $y(x)$ is replaced by $x(t)$, and b is replaced by $-b$. Using the result of Exercise 21(b), we obtain, since $x(0) = 0$,

$$\begin{aligned}
x(t) &= \left(x(0) - \frac{a}{b}\right)e^{-bt} + \frac{a}{b}\\
&= \frac{a}{b}\left(1 - e^{-bt}\right).
\end{aligned}$$

c) We will have $x(t) = \frac{1}{2}(a/b)$ if $1 - e^{-bt} = \frac{1}{2}$, that is, if $e^{-bt} = \frac{1}{2}$, or $-bt = \ln(1/2) = -\ln 2$. The time required to attain half the limiting concentration is $t = (\ln 2)/b$.

25. Let $T(t)$ be the reading t minutes after the Thermometer is moved outdoors. Thus $T(0) = 72$, $T(1) = 48$.
By Newton's law of cooling, $\dfrac{dT}{dt} = k(T - 20)$.
If $V(t) = T(t) - 20$, then $\dfrac{dV}{dt} = kV$, so $V(t) = V(0)e^{kt} = 52e^{kt}$.
Also $28 = V(1) = 52e^k$, so $k = \ln(7/13)$.
Thus $V(5) = 52e^{5\ln(7/13)} \approx 2.354$. At $t = 5$ the thermometer reads about $T(5) = 20 + 2.354 = 22.35\,°\mathrm{C}$.

26. Let $T(t)$ be the temperature of the object t minutes after its temperature was $45\,°$ C. Thus $T(0) = 45$ and $T(40) = 20$. Also $\dfrac{dT}{dt} = k(T + 5)$. Let $u(t) = T(t) + 5$, so $u(0) = 50$, $u(40) = 25$, and $\dfrac{du}{dt} = \dfrac{dT}{dt} = k(T + 5) = ku$. Thus,

$$\begin{aligned}
u(t) &= 50e^{kt},\\
25 = u(40) &= 50e^{40k},\\
\Rightarrow k &= \frac{1}{40}\ln\frac{25}{50} = \frac{1}{40}\ln\frac{1}{2}.
\end{aligned}$$

We wish to know t such that $T(t) = 0$, i.e., $u(t) = 5$, hence

$$\begin{aligned}
5 &= u(t) = 50e^{kt}\\
t &= \frac{40\ln\left(\frac{5}{50}\right)}{\ln\left(\frac{1}{2}\right)} = 132.88\text{ min}.
\end{aligned}$$

Hence, it will take about $(132.88 - 40) = 92.88$ minutes more to cool to $0\,°$ C.

27. Let $T(t)$ be the temperature of the body t minutes after it was 5°.
Thus $T(0) = 5$, $T(4) = 10$. Room temperature = 20°.
By Newton's law of cooling (warming) $\frac{dT}{dt} = k(T - 20)$.
If $V(t) = T(t) - 20$ then $\frac{dV}{dt} = kV$,
so $V(t) = V(0)e^{kt} = -15e^{kt}$.
Also $-10 = V(4) = -15e^{4k}$, so $k = \frac{1}{4}\ln\left(\frac{2}{3}\right)$.
If $T(t) = 15°$, then $-5 = V(t) = -15e^{kt}$

$$\text{so } t = \frac{1}{k}\ln\left(\frac{1}{3}\right) = 4\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{2}{3}\right)} \approx 10.838.$$

It will take a further 6.84 minutes to warm to 15 °C.

28. By the solution given for the logistic equation, we have

$$y_1 = \frac{Ly_0}{y_0 + (L - y_0)e^{-k}}, \qquad y_2 = \frac{Ly_0}{y_0 + (L - y_0)e^{-2k}}$$

Thus $y_1(L - y_0)e^{-k} = (L - y_1)y_0$, and
$y_2(L - y_0)e^{-2k} = (L - y_2)y_0$.
Square the first equation and thus eliminate e^{-k}:

$$\left(\frac{(L - y_1)y_0}{y_1(L - y_0)}\right)^2 = \frac{(L - y_2)y_0}{y_2(L - y_0)}$$

Now simplify: $y_0y_2(L - y_1)^2 = y_1^2(L - y_0)(L - y_2)$
$y_0y_2L^2 - 2y_1y_0y_2L + y_0y_1^2y_2 = y_1^2L^2 - y_1^2(y_0 + y_2)L + y_0y_1^2y_2$

$$\text{Assuming } L \neq 0, \quad L = \frac{y_1^2(y_0 + y_2) - 2y_0y_1y_2}{y_1^2 - y_0y_2}.$$

If $y_0 = 3$, $y_1 = 5$, $y_2 = 6$, then

$$L = \frac{25(9) - 180}{25 - 18} = \frac{45}{7} \approx 6.429.$$

29. The rate of growth of y in the logistic equation is

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Since

$$\frac{dy}{dt} = -\frac{k}{L}\left(y - \frac{L}{2}\right)^2 + \frac{kL}{4},$$

thus $\frac{dy}{dt}$ is greatest when $y = \frac{L}{2}$.

30. The solution $y = \frac{Ly_0}{y_0 + (L - y_0)e^{-kt}}$ is valid on the largest interval containing $t = 0$ on which the denominator does not vanish.
If $y_0 > L$ then $y_0 + (L - y_0)e^{-kt} = 0$ if
$t = t^* = -\frac{1}{k}\ln\frac{y_0}{y_0 - L}$.
Then the solution is valid on (t^*, ∞).
$\lim_{t \to t^*+} y(t) = \infty$.

31. The solution

$$y = \frac{Ly_0}{y_0 + (L - y_0)e^{-kt}}$$

of the logistic equation is valid on any interval containing $t = 0$ and not containing any point where the denominator is zero. The denominator is zero if $y_0 = (y_0 - L)e^{-kt}$, that is, if

$$t = t^* = -\frac{1}{k}\ln\left(\frac{y_0}{y_0 - L}\right).$$

Assuming k and L are positive, but y_0 is negative, we have $t^* > 0$. The solution is therefore valid on $(-\infty, t^*)$. The solution approaches $-\infty$ as $t \to t^*-$.

32.

$$\begin{aligned} y(t) &= \frac{L}{1 + Me^{-kt}} \\ 200 = y(0) &= \frac{L}{1 + M} \\ 1,000 = y(1) &= \frac{L}{1 + Me^{-k}} \\ 10,000 &= \lim_{t\to\infty} y(t) = L \end{aligned}$$

Thus $200(1 + M) = L = 10,000$, so $M = 49$. Also $1,000(1 + 49e^{-k}) = L = 10,000$, so $e^{-k} = 9/49$ and $k = \ln(49/9) \approx 1.695$.

33.

$$y(3) = \frac{L}{1 + Me^{-3k}} = \frac{10,000}{1 + 49(9/49)^3} \approx 7671 \text{ cases}$$

$$y'(3) = \frac{LkMe^{-3k}}{(1 + Me^{-3k})^2} \approx 3,028 \text{ cases/week.}$$

Section 3.5 The Inverse Trigonometric Functions (page 197)

1. $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$
2. $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$
3. $\tan^{-1}(-1) = -\frac{\pi}{4}$
4. $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$
5. $\sin(\sin^{-1} 0.7) = 0.7$
6. $\cos(\sin^{-1} 0.7) = \sqrt{1 - \sin^2(\arcsin 0.7)} = \sqrt{1 - 0.49} = \sqrt{0.51}$
7. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$
8. $\sin^{-1}(\cos 40°) = 90° - \cos^{-1}(\cos 40°) = 50°$
9. $\cos^{-1}\left(\sin(-0.2)\right) = \frac{\pi}{2} - \sin^{-1}\left(\sin(-0.2)\right) = \frac{\pi}{2} + 0.2$

10. $\sin\left(\cos^{-1}(-\frac{1}{3})\right) = \sqrt{1-\cos^2(\arccos(-\frac{1}{3})}$
$= \sqrt{1-\frac{1}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

11. $\cos\left(\tan^{-1}\frac{1}{2}\right) = \frac{1}{\sec\left(\tan^{-1}\frac{1}{2}\right)}$
$= \frac{1}{\sqrt{1+\tan^2\left(\tan^{-1}\frac{1}{2}\right)}} = \frac{2}{\sqrt{5}}$

12. $\tan(\tan^{-1} 200) = 200$

13. $\sin(\cos^{-1} x) = \sqrt{1-\cos^2(\cos^{-1} x)}$
$= \sqrt{1-x^2}$

14. $\cos(\sin^{-1} x) = \sqrt{1-\sin^2(\sin^{-1} x)} = \sqrt{1-x^2}$

15. $\cos(\tan^{-1} x) = \frac{1}{\sec(\tan^{-1} x)} = \frac{1}{\sqrt{1+x^2}}$

16. $\tan(\arctan x) = x \Rightarrow \sec(\arctan x) = \sqrt{1+x^2}$
$\Rightarrow \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$
$\Rightarrow \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$

17. $\tan(\cos^{-1} x) = \frac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x}$
$= \frac{\sqrt{1-x^2}}{x}$ (by # 13)

18. $\cos(\sec^{-1}x) = \frac{1}{x} \Rightarrow \sin(\sec^{-1}x) = \sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{|x|}$
$\Rightarrow \tan(\sec^{-1}x) = \sqrt{x^2-1}\,\operatorname{sgn} x$
$= \begin{cases} \sqrt{x^2-1} & \text{if } x \ge 1 \\ -\sqrt{x^2-1} & \text{if } x \le -1 \end{cases}$

19. $y = \sin^{-1}\left(\frac{2x-1}{3}\right)$
$y' = \frac{1}{\sqrt{1-\left(\frac{2x-1}{3}\right)^2}}\frac{2}{3}$
$= \frac{2}{\sqrt{9-(4x^2-4x+1)}}$
$= \frac{1}{\sqrt{2+x-x^2}}$

20. $y = \tan^{-1}(ax+b), \qquad y' = \frac{a}{1+(ax+b)^2}.$

21. $y = \cos^{-1}\frac{x-b}{a}$
$y' = -\frac{1}{\sqrt{1-\frac{(x-b)^2}{a^2}}}\frac{1}{a}$
$= \frac{-1}{\sqrt{a^2-(x-b)^2}}$ (assuming) $a > 0$).

22. $f(x) = x\sin^{-1} x$
$f'(x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}.$

23. $f(t) = t\tan^{-1} t$
$f'(t) = \tan^{-1} t + \frac{t}{1+t^2}$

24. $u = z^2\sec^{-1}(1+z^2)$
$\frac{du}{dz} = 2z\sec^{-1}(1+z^2) + \frac{z^2(2z)}{(1+z^2)\sqrt{(1+z^2)^2-1}}$
$= 2z\sec^{-1}(1+z^2) + \frac{2z^2\operatorname{sgn}(z)}{(1+z^2)\sqrt{z^2+2}}$

25. $F(x) = (1+x^2)\tan^{-1} x$
$F'(x) = 2x\tan^{-1} x + 1$

26. $y = \sin^{-1}\left(\frac{a}{x}\right) \qquad (|x| > |a|)$
$y' = \frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\left[-\frac{a}{x^2}\right] = -\frac{a}{|x|\sqrt{x^2-a^2}}$

27. $G(x) = \frac{\sin^{-1} x}{\sin^{-1}(2x)}$
$G'(x) = \frac{\sin^{-1}(2x)\frac{1}{\sqrt{1-x^2}} - \sin^{-1} x\frac{2}{\sqrt{1-4x^2}}}{\left(\sin^{-1}(2x)\right)^2}$
$= \frac{\sqrt{1-4x^2}\sin^{-1}(2x) - 2\sqrt{1-x^2}\sin^{-1} x}{\sqrt{1-x^2}\sqrt{1-4x^2}\left(\sin^{-1}(2x)\right)^2}$

28. $H(t) = \frac{\sin^{-1} t}{\sin t}$
$H'(t) = \frac{\sin t\left(\frac{1}{\sqrt{1-t^2}}\right) - \sin^{-1} t\cos t}{\sin^2 t}$
$= \frac{1}{(\sin t)\sqrt{1-t^2}} - \csc t\cot t\sin^{-1} t$

29. $f(x) = (\sin^{-1} x^2)^{1/2}$
$f'(x) = \frac{1}{2}(\sin^{-1} x^2)^{-1/2}\frac{2x}{\sqrt{1-x^4}}$
$= \frac{x}{\sqrt{1-x^4}\sqrt{\sin^{-1} x^2}}$

30. $y = \cos^{-1}\left(\dfrac{a}{\sqrt{a^2+x^2}}\right)$

$$y' = -\left(1-\frac{a^2}{a^2+x^2}\right)^{-1/2}\left[-\frac{a}{2}(a^2+x^2)^{-3/2}(2x)\right] = \frac{a\,\mathrm{sgn}\,(x)}{a^2+x^2}$$

31. $y = \sqrt{a^2-x^2} + a\sin^{-1}\dfrac{x}{a}$

$$y' = -\frac{x}{\sqrt{a^2-x^2}} + \frac{a}{\sqrt{1-\dfrac{x^2}{a^2}}}\frac{1}{a} = \frac{a-x}{\sqrt{a^2-x^2}} = \sqrt{\frac{a-x}{a+x}} \qquad (a>0)$$

32. $y = a\cos^{-1}\left(1-\dfrac{x}{a}\right) - \sqrt{2ax-x^2} \qquad (a>0)$

$$y' = -a\left[1-\left(1-\frac{x}{a}\right)^2\right]^{-1/2}\left(-\frac{1}{a}\right) - \frac{2a-2x}{2\sqrt{2ax-x^2}} = \frac{x}{\sqrt{2ax-x^2}}$$

33. $\tan^{-1}\left(\dfrac{2x}{y}\right) = \dfrac{\pi x}{y^2}$

$$\frac{1}{1+\dfrac{4x^2}{y^2}}\,\frac{2y-2xy'}{y^2} = \pi\frac{y^2-2xyy'}{y^4}$$

At (1, 2) $\dfrac{1}{2}\dfrac{4-2y'}{4} = \pi\dfrac{4-4y'}{16}$

$8-4y' = 4\pi - 4\pi y' \Rightarrow y' = \dfrac{\pi-2}{\pi-1}$

At (1, 2) the slope is $\dfrac{\pi-2}{\pi-1}$

34. If $y = \sin^{-1}x$, then $y' = \dfrac{1}{\sqrt{1-x^2}}$. If the slope is 2 then $\dfrac{1}{\sqrt{1-x^2}} = 2$ so that $x = \pm\dfrac{\sqrt{3}}{2}$. Thus the equations of the two tangent lines are

$$y = \frac{\pi}{3} + 2\left(x-\frac{\sqrt{3}}{2}\right) \text{ and } y = -\frac{\pi}{3} + 2\left(x+\frac{\sqrt{3}}{2}\right).$$

35. $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}} > 0$ on $(-1, 1)$.
Therefore, $\sin^{-1}$ is increasing.
$\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2} > 0$ on $(-\infty, \infty)$.
Therefore $\tan^{-1}$ is increasing.
$\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}} < 0$ on $(-1, 1)$.
Therefore $\cos^{-1}$ is decreasing.

36. Since the domain of $\sec^{-1}$ consists of two disjoint intervals $(-\infty, -1]$ and $[1, \infty)$, the fact that the derivative of $\sec^{-1}$ is positive wherever defined does not imply that $\sec^{-1}$ is increasing over its whole domain, only that it is increasing on each of those intervals taken independently. In fact, $\sec^{-1}(-1) = \pi > 0 = \sec^{-1}(1)$ even though $-1 < 1$.

37.
$$\frac{d}{dx}\csc^{-1}x = \frac{d}{dx}\sin^{-1}\frac{1}{x} = \frac{1}{\sqrt{1-\dfrac{1}{x^2}}}\left(-\frac{1}{x^2}\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

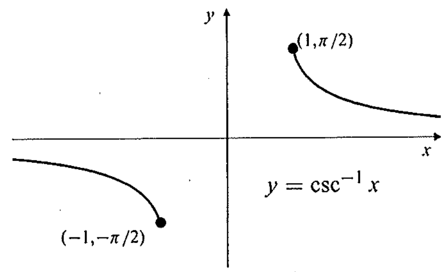

Fig. 3.5.37

38. $\cot^{-1}x = \arctan(1/x)$;

$$\frac{d}{dx}\cot^{-1}x = \frac{1}{1+\dfrac{1}{x^2}}\,\frac{-1}{x^2} = -\frac{1}{1+x^2}$$

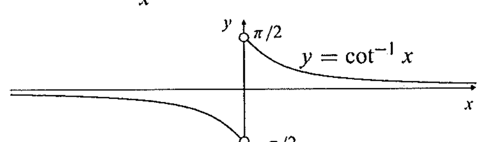

Fig. 3.5.38

Remark: the domain of $\cot^{-1}$ can be extended to include 0 by defining, say, $\cot^{-1}0 = \pi/2$. This will make $\cot^{-1}$ right-continuous (but not continuous) at $x = 0$. It is also possible to define $\cot^{-1}$ in such a way that it is continuous on the whole real line, but we would then lose the identity $\cot^{-1}x = \tan^{-1}(1/x)$, which we prefer to maintain for calculation purposes.

39. $\frac{d}{dx}(\tan^{-1}x+\cot^{-1}x)=\frac{d}{dx}\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)$

$$=\frac{1}{1+x^2}+\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)=0 \text{ if } x\neq 0$$

Thus $\tan^{-1}x+\cot^{-1}x=C_1$ (const. for $x>0$)
At $x=1$ we have $\frac{\pi}{4}+\frac{\pi}{4}=C_1$
Thus $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ for $x>0$.
Also $\tan^{-1}x+\cot^{-1}x=C_2$ for $(x<0)$.
At $x=-1$, we get $-\frac{\pi}{4}-\frac{\pi}{4}=C_2$.
Thus $\tan^{-1}x+\cot^{-1}x=-\frac{\pi}{2}$ for $x<0$.

40. If $g(x)=\tan(\tan^{-1}x)$ then

$$g'(x)=\frac{\sec^2(\tan^{-1}x)}{1+x^2}=\frac{1+[\tan(\tan^{-1}x)]^2}{1+x^2}=\frac{1+x^2}{1+x^2}=1.$$

If $h(x)=\tan^{-1}(\tan x)$ then h is periodic with period π, and

$$h'(x)=\frac{\sec^2 x}{1+\tan^2 x}=1$$

provided that $x\neq(k+\frac{1}{2})\pi$ where k is an integer. $h(x)$ is not defined at odd multiples of $\frac{\pi}{2}$.

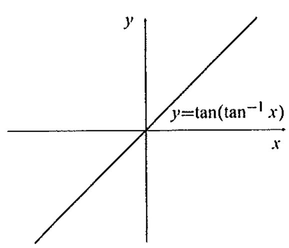

Fig. 3.5.40(a)

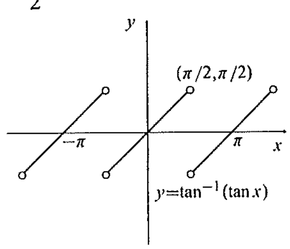

Fig. 3.5.40(b)

41. $\frac{d}{dx}\cos^{-1}(\cos x)=\frac{-1}{\sqrt{1-\cos^2 x}}(-\sin x)$

$$=\begin{cases}1 & \text{if } \sin x>0\\ -1 & \text{if } \sin x<0\end{cases}$$

$\cos^{-1}(\cos x)$ is continuous everywhere and differentiable everywhere except at $x=n\pi$ for integers n.

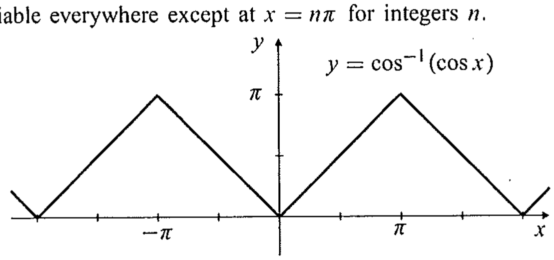

Fig. 3.5.41

42. $\frac{d}{dx}\sin^{-1}(\cos x)=\frac{1}{\sqrt{1-\cos^2 x}}(-\sin x)$

$$=\begin{cases}-1 & \text{if } \sin x>0\\ 1 & \text{if } \sin x<0\end{cases}$$

$\sin^{-1}(\cos x)$ is continuous everywhere and differentiable everywhere except at $x=n\pi$ for integers n.

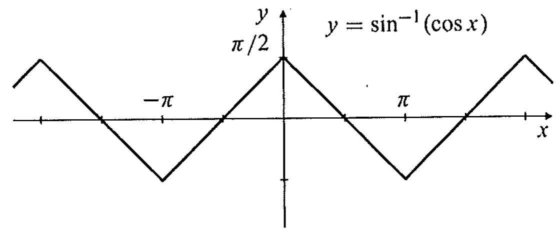

Fig. 3.5.42

43. $\frac{d}{dx}\tan^{-1}(\tan x)=\frac{1}{1+\tan^2 x}(\sec^2 x)=1$ except at odd multiples of $\pi/2$.

$\tan^{-1}(\tan x)$ is continuous and differentiable everywhere except at $x=(2n+1)\pi/2$ for integers n. It is not defined at those points.

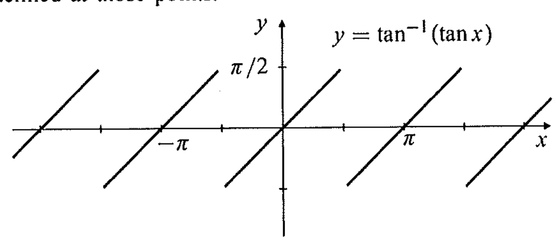

Fig. 3.5.43

44. $\frac{d}{dx}\tan^{-1}(\cot x)=\frac{1}{1+\cot^2 x}(-\csc^2 x)=-1$ except at integer multiples of π.

$\tan^{-1}(\cot x)$ is continuous and differentiable everywhere except at $x=n\pi$ for integers n. It is not defined at those points.

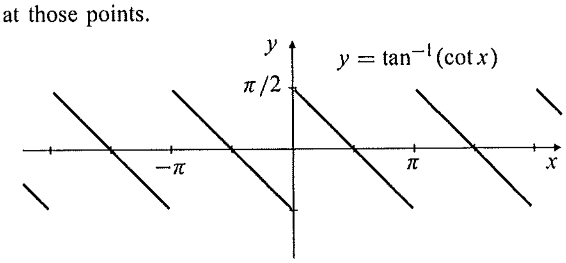

Fig. 3.5.44

11. Let the length, width, depth, and volume at time t be l, w, h and V respectively. Thus $V = lwh$, and

$$\frac{dV}{dt} = \frac{dl}{dt}wh + lh\frac{dw}{dt} + lw\frac{dh}{dt}.$$

If $l = 6$ cm, w =5cm, h= 4cm, $\frac{dl}{dt} = \frac{dh}{dt} = 1$m/s, and $\frac{dw}{dt} = -2$cm/s, then

$$\frac{dV}{dt} = 20 - 48 + 30 = 2.$$

The volume is increasing at a rate of 2 cm^3/s.

12. Let the length, width and area at time t be x, y and A respectively. Thus $A = xy$ and

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}.$$

If $\frac{dA}{dt} = 5$, $\frac{dx}{dt} = 10$, $x = 20$, $y = 16$, then

$$5 = 20\frac{dy}{dt} + 16(10) \Rightarrow \frac{dy}{dt} = -\frac{31}{4}.$$

Thus, the width is decreasing at $\frac{31}{4}$ m/s.

13. $y = x^2$. Thus $\frac{dy}{dt} = 2x\frac{dx}{dt}$. If $x = -2$ and $\frac{dx}{dt} = -3$, then $\frac{dy}{dt} = -4(-3) = 12$. y is increasing at rate 12.

14. Since $x^2y^3 = 72$, then

$$2xy^3\frac{dx}{dt} + 3x^2y^2\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{2y}{3x}\frac{dx}{dt}.$$

If $x = 3$, $y = 2$, $\frac{dx}{dt} = 2$, then $\frac{dy}{dt} = -\frac{8}{9}$. Hence, the vertical velocity is $-\frac{8}{9}$ units/s.

15. We have

$$xy = t \Rightarrow x\frac{dy}{dt} + y\frac{dx}{dt} = 1$$
$$y = tx^2 \Rightarrow \frac{dy}{dt} = x^2 + 2xt\frac{dx}{dt}$$

At $t = 2$ we have $xy = 2$, $y = 2x^2 \Rightarrow 2x^3 = 2 \Rightarrow x = 1$, $y = 2$.
Thus $\frac{dy}{dt} + 2\frac{dx}{dt} = 1$, and $1 + 4\frac{dx}{dt} = \frac{dy}{dt}$.
So $1 + 6\frac{dx}{dt} = 1 \Rightarrow \frac{dx}{dt} = 0 \Rightarrow \frac{dy}{dt} = 1 \Rightarrow$.
Distance D from origin satisfies $D = \sqrt{x^2 + y^2}$. So

$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{1}{\sqrt{5}}\Big(1(0) + 2(1)\Big) = \frac{2}{\sqrt{5}}.$$

The distance from the origin is increasing at a rate of $2/\sqrt{5}$.

16. From the figure, $x^2 + k^2 = s^2$. Thus

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

When angle $PCA = 45°$, $x = k$ and $s = \sqrt{2}k$. The radar gun indicates that $ds/dt = 100$ km/h. Thus $dx/dt = 100\sqrt{2}k/k \approx 141$. The car is travelling at about 141 km/h.

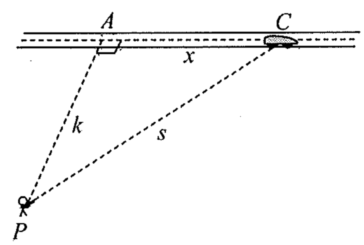

Fig. 4.1.16

17. We continue the notation of Exercise 16. If $dx/dt = 90$ km/h, and angle $PCA = 30°$, then $s = 2k$, $x = \sqrt{3}k$, and $ds/dt = (\sqrt{3}k/2k)(90) = 45\sqrt{3} = 77.94$. The radar gun will read about 78 km/h.

18. Let the distances x and y be as shown at time t. Thus $x^2 + y^2 = 25$ and $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.
If $\frac{dx}{dt} = \frac{1}{3}$ and $y = 3$, then $x = 4$ and $\frac{4}{3} + 3\frac{dy}{dt} = 0$ so $\frac{dy}{dt} = -\frac{4}{9}$.
The top of the ladder is slipping down at a rate of $\frac{4}{9}$ m/s.

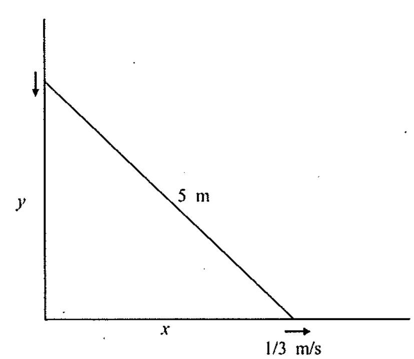

Fig. 4.1.18

19. Let x and y be the distances shown in the following figure. From similar triangles:

$$\frac{x}{2} = \frac{x+y}{5} \Rightarrow x = \frac{2y}{3} \Rightarrow \frac{dx}{dt} = \frac{2}{3}\frac{dy}{dt}.$$

Since $\dfrac{dy}{dt} = -\dfrac{1}{2}$, then

$$\frac{dx}{dt} = -\frac{1}{3} \text{ and } \frac{d}{dt}(x+y) = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}.$$

Hence, the man's shadow is decreasing at $\frac{1}{3}$ m/s and the shadow of his head is moving towards the lamppost at a rate of $\frac{5}{6}$ m/s.

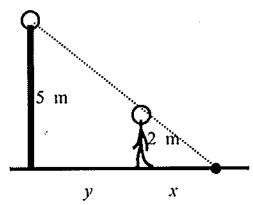

Fig. 4.1.19

20.

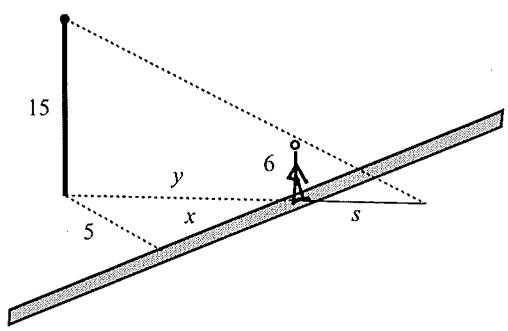

Fig. 4.1.20

Refer to the figure. s, y, and x are, respectively, the length of the woman's shadow, the distances from the woman to the lamppost, and the distances from the woman to the point on the path nearest the lamppost. From one of triangles in the figure we have

$$y^2 = x^2 + 25.$$

If $x = 12$, then $y = 13$. Moreover,

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

We are given that $dx/dt = 2$ ft/s, so $dy/dt = 24/13$ ft/s when $x = 12$ ft. Now the similar triangles in the figure show that

$$\frac{s}{6} = \frac{s+y}{15},$$

so that $s = 2y/3$. Hence $ds/dt = 48/39$. The woman's shadow is changing at rate 48/39 ft/s when she is 12 ft from the point on the path nearest the lamppost.

21.

$$C = 10,000 + 3x + \frac{x^2}{8,000}$$

$$\frac{dC}{dt} = \left(3 + \frac{x}{4,000}\right)\frac{dx}{dt}.$$

If $dC/dt = 600$ when $x = 12,000$, then $dx/dt = 100$. The production is increasing at a rate of 100 tons per day.

22. Let x, y be distances travelled by A and B from their positions at 1:00 pm in t hours.

Thus $\dfrac{dx}{dt} = 16$ km/h, $\dfrac{dy}{dt} = 20$ km/h.

Let s be the distance between A and B at time t.

Thus $s^2 = x^2 + (25+y)^2$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2(25+y)\frac{dy}{dt}$$

At 1:30 $\left(t = \frac{1}{2}\right)$ we have $x = 8$, $y = 10$,
$s = \sqrt{8^2 + 35^2} = \sqrt{1289}$ so

$$\sqrt{1289}\frac{ds}{dt} = 8 \times 16 + 35 \times 20 = 828$$

and $\dfrac{ds}{dt} = \dfrac{828}{\sqrt{1289}} \approx 23.06$. At 1:30, the ships are separating at about 23.06 km/h.

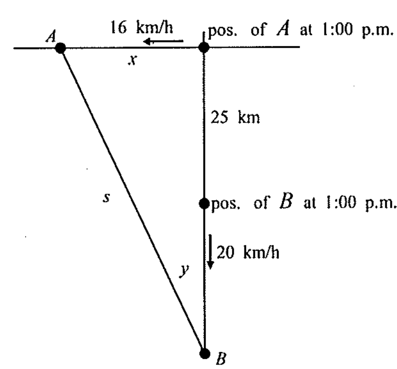

Fig. 4.1.22

23. Let θ and ω be the angles that the minute hand and hour hand made with the vertical t minutes after 3 o'clock. Then

$$\frac{d\theta}{dt} = \frac{\pi}{30} \text{ rad/min}$$

$$\frac{d\omega}{dt} = \frac{\pi}{360} \text{ rad/min}.$$