20. For $x^2 = 0$ we have $x_{n+1} = x_n - (x_n^2/(2x_n)) = x_n/2$. If $x_0 = 1$, then $x_1 = 1/2$, $x_2 = 1/4$, $x_3 = 1/8$.

a) $x_n = 1/2^n$, by induction.

b) x_n approximates the root $x = 0$ to within 0.0001 provided $2^n > 10,000$. We need $n \geq 14$ to ensure this.

c) To ensure that x_n^2 is within 0.0001 of 0 we need $(1/2^n)^2 < 0.0001$, that is, $2^{2n} > 10,000$. We need $n \geq 7$.

d) Convergence of Newton approximations to the root $x = 0$ of $x^2 = 0$ is slower than usual because the derivative $2x$ of x^2 is zero at the root.

21. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$,
$f'(x) = \begin{cases} 1/(2\sqrt{x}) & \text{if } x > 0 \\ -1/(2\sqrt{-x}) & \text{if } x < 0 \end{cases}$.
The Newton's Method formula says that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - 2x_n = -x_n.$$

If $x_0 = a$, then $x_1 = -a$, $x_2 = a$, and, in general, $x_n = (-1)^n a$. The approximations oscillate back and forth between two numbers.
If one observed that successive approximations were oscillating back and forth between two values a and b, one should try their average, $(a + b)/2$, as a new starting guess. It may even turn out to be the root!

22. Newton's Method formula for $f(x) = x^{1/3}$ is

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{(1/3)x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

If $x_0 = 1$, then $x_1 = -2$, $x_2 = 4$, $x_3 = -8$, $x_4 = 16$, and, in general, $x_n = (-2)^n$. The successive "approximations" oscillate ever more widely, diverging from the root at $x = 0$.

23. Newton's Method formula for $f(x) = x^{2/3}$ is

$$x_{n+1} = x_n - \frac{x_n^{2/3}}{(2/3)x_n^{-1/3}} = x_n - \tfrac{3}{2}x_n = -\tfrac{1}{2}x_n.$$

If $x_0 = 1$, then $x_1 = -1/2$, $x_2 = 1/4$, $x_3 = -1/8$, $x_4 = 1/16$, and, in general, $x_n = (-1/2)^n$. The successive approximations oscillate around the root $x = 0$, but still converge to it (though more slowly than is usual for Newton's Method).

24. Since $x_{n+1} = \dfrac{x_n^2 - 1}{2x_n}$, we have

$$1 + x_{n+1}^2 = 1 + \left(\frac{x_n^2 - 1}{2x_n}\right)^2 = \left(\frac{x_n^2 + 1}{2x_n}\right)^2.$$

It follows that

$$y_{n+1} = \left(\frac{2x_n}{x_n^2 + 1}\right)^2 = 4y_n^2 \frac{1 - y_n}{y_n} = 4y_n(1 - y_n).$$

25. Let $y_j = \sin^2(u_j)$.

a) Since
$\sin^2(u_{n+1}) = 4\sin^2(u_n)(1 - \sin^2(u_n)) = 4\sin^2(u_n)\cos^2(u_n) = \sin^2(2u_n)$, we have $u_{n+1} = 2u_n$. Thus $u_{n+1} = 2^n u_0$. It follows that

$$dy_n = 2\sin(u_n)\cos(u_n)\,2^n\,du_0.$$

b) Since $y_n = \dfrac{1}{1 + x_n^2}$, we have

$$dy_n = -\frac{2x_n}{(1 + x_n^2)^2}\,dx_n.$$

Hence

$$dx_n = -\frac{(1 + x_n^2)^2}{2x_n}\,2\sin(u_n)\cos(u_n)\,2^n\,du_0.$$

Since the values of x_n are assumed to neither converge nor diverge, the exponential factor 2^n will dominate for large n

26. Let $g(x) = f(x) - x$ for $a \leq x \leq b$. g is continuous (because f is), and since $a \leq f(x) \leq b$ whenever $a \leq x \leq b$ (by condition (i)), we know that $g(a) \geq 0$ and $g(b) \leq 0$. By the Intermediate-Value Theorem there exists r in $[a, b]$ such that $g(r) = 0$, that is, such that $f(r) = r$. The fixed point r is unique because if there were two such fixed points, say r_1 and r_2, then condition (ii) would imply that

$$|r_1 - r_2| = |f(r_1) - f(r_2)| \leq K|r_1 - r_2|,$$

which is impossible if $r_1 \neq r_2$ and $K < 1$.

27. We are given that there is a constant K satisfying $0 < K < 1$, such that

$$|f(u) - f(v)| \leq K|u - v|$$

holds whenever u and v are in $[a, b]$. Pick any x_0 in $[a, b]$, and let $x_1 = f(x_0)$, $x_2 = f(x_1)$, and, in general, $x_{n+1} = f(x_n)$. Let r be the fixed point of f in $[a, b]$ found in Exercise 24. Thus $f(r) = r$. We have

$$|x_1 - r| = |f(x_0) - f(r)| \leq K|x_0 - r|$$
$$|x_2 - r| = |f(x_1) - f(r)| \leq K|x_1 - r| \leq K^2|x_0 - r|,$$