SOLUTIONS

# Math 120 Midterm 2

Nov 13, 2013

Duration: 50 minutes

Name: ______________________ Student Number: ______________________

**This exam should have 9 pages. No textbooks, calculators, or other aids are allowed. There are 5 problems in this exam: problem 1 is 18 points and the remaining problems are 8 points each (50 points total).**

## Problem 1 (18 points)

In this problem, each part is 3 points. Partial credit will be given, so show your work.

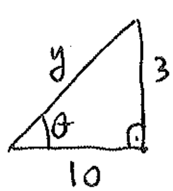
(a) Calculate $\lim_{x\to 0} \frac{\sin(3x)}{x\cos(x)+\sin(x)}$. $= L$

(provided both limits on the RHS exist)

$$L = \lim_{x\to 0} \frac{\sin 3x}{x\left[\cos x + \frac{\sin x}{x}\right]} \overset{\downarrow}{=} \lim_{x\to 0} \frac{\sin 3x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x + \frac{\sin x}{x}}$$

$$= \lim_{x\to 0} 3 \cdot \frac{\sin 3x}{3x} \cdot \frac{1}{\lim_{x\to 0} \cos x + \lim_{x\to 0} \frac{\sin x}{x}} = 3 \cdot \frac{1}{2} = \frac{3}{2} //$$

(b) Calculate $\sin(\arctan(0.3)) = \theta$. Since $\tan\theta = 0.3$ and $\theta \in \operatorname{ran}(\arctan) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we know that $0 < \theta < \frac{\pi}{2}$. Then:




$$y = \sqrt{100+9} = \sqrt{109}$$

$$\sin\theta = \frac{3}{y} = \frac{3}{\sqrt{109}} = \frac{3\sqrt{109}}{109} //$$

(c) Calculate $f^{(40)}(x)$ if $f(x) = \sin(3x + 5)$.

Set $g(x) = \sin x$ ; $h(x) = 3x+5$. Then $f(x) = g(h(x))$.

$\Rightarrow \quad f'(x) = g'(h(x)) \cdot h'(x) = 3 \cdot g'(h(x))$

$f^2(x) = 3^2 g''(h(x)), \ldots \quad f^{(n)} = 3^n g^{(n)}(h(x))$.

With $g(x) = \sin x$, we know $g^{(4)}(x) = \sin x \Rightarrow g^{(4k)}(x) = \sin x$ for all positive integers $k$. Thus:

$\cancel{f^{(40)}(x) = 3^{40} g^{(40)}(h(x)) = 3^{40} \sin}$

$f^{(40)}(x) = 3^{40} g^{(40)}(h(x)) = 3^{40} \cdot \sin(h(x)) = 3^{40} \sin(3x+5)$ //.

(d) Let $y = f(x)$ and $z = g(x)$ satisfy

$$e^y z^2 + zy \log|z| = 0.$$

Calculate $f'(1)$ if $f(1) = 0$, $g(1) = 1$, and $g'(1) = 2$.

Differentiate implicitly:

$e^y \cdot y' \cdot z^2 + e^y \cdot 2z \cdot z' + y' z \log|z| + y(\log|z| + 1) z' = 0$.

Substituting $y = 0$, $z = 1$, $z' = 2$, we get

$y' + 4 = 0 \quad \Rightarrow \quad y' = -4$ //

(e) Let $f(x) = x^{sin^2(x)}$. Calculate $f'(x)$.

Set $y = x^{\sin^2 x} \Rightarrow \ln y = \sin^2 x \ln x$

$\Rightarrow \frac{y'}{y} = 2\sin x \cos x \ln x + \frac{\sin^2 x}{x}$

So: $y' = x^{\sin^2 x} \cdot \left( 2 \sin x \cos x \ln x + \frac{\sin^2 x}{x} \right)$

$= x^{\sin^2 x} \left( \sin 2x \ln x + \frac{\sin^2 x}{x} \right)$ //

(f) Solve the initial value problem given by

$$y'(t) = 2(5 - y(t)); \quad y(0) = 15.$$

Rewrite: $y' = -2(y-5)$

Set $u = y - 5 \Rightarrow u' = y'$, & $y = u + 5$

So: $u' = -2u \Rightarrow u(t) = C \cdot e^{-2t} \Rightarrow y(t) = C \cdot e^{-2t} + 5$

$y(0) = C + 5 = 15 \Rightarrow C = 10$

So: $y(t) = 10 e^{-2t} + 5$ //

**Problem 2 (8 points)**

Bacteria grow in a certain culture at a rate proportional to the present number of bacteria such that the doubling time for the population is 4 days. Suppose the population grows by 8000 between the end of day 4 and the end of day 8.

(a) What was the initial population?

(b) What will the population be at the end of day 10?

(a) $y(t) = C \cdot e^{kt}$.

Doubling time = 4 days: $y(t+4) = 2y(t) \quad \forall t$.

$\Leftrightarrow \cancel{C} \cdot e^{k(t+4)} = 2\cancel{C} e^{kt} \Leftrightarrow \cancel{e^{kt}} \cdot e^{k \cdot 4} = 2\cancel{e^{kt}}$

So, $4k = \ln 2 \Rightarrow k = \frac{\ln 2}{4}$

Accordingly: $y(t) = C \cdot e^{\ln 2 \cdot t/4} = C \cdot 2^{t/4}$.

Now: $y(8) - y(4) = C \cdot 2^2 - C \cdot 2^1 = 2C = 8000$

$\Rightarrow C = 4000 = y(0)$ //

(b) From above: $y(t) = 4000 \cdot 2^{t/4}$ (t in days)

So, $y(10) = 4000 \cdot 2^{10/4} = 4000 \cdot 2^2 \cdot \sqrt{2} = 16{,}000\sqrt{2}$

**Problem 3 (8 points)**

Let $f(x)$ be a function that is differentiable on $\mathbb{R}$. Suppose that $f'(x) \geq 0$ for all $x$ with $f'(x) = 0$ if and only if $x = x_1$ (i.e., the derivative vanishes at a single point).

(a) Give an example of such a function.

(b) Prove that $f$ is strictly increasing on $[x_1, \infty)$.

(a) $f(x) = x^3$ (or $f(x) = x^n$ with any odd integer $n > 0$).

satisfies $f'(x) = 3x^2 = 0 \iff x = 0$

(b) Let $a, b \in [x_1, \infty)$ with $a < b$. Then $b > x_1$ and by MVT $\exists\, c \in (a,b)$

s.t. $\dfrac{f(b) - f(a)}{b - a} = f'(c)$. Since $c > a \geq x_1$, we know that

$f'(c) > 0$ (as $f'(x) \geq 0$ $\forall x$ with $f'(x) = 0 \iff x = x_1$).

Also $b - a > 0$, as $b > a$. Thus

$f(b) - f(a) = f'(c)(b-a) > 0 \implies f(b) > f(a)$.

We conclude that $f$ is increasing on $[x_1, \infty)$.

(NOTE: in the above argument, $a$ can be equal to $x_1$).

**Problem 4 (8 points)**

The equation $x^2 + 7xy + y^2 = 9$ describes a rotated and translated hyperbola.

(a) Find $y'$ and $y''$ at the point $(1, 1)$.

(b) Find all points on the hyperbola where the tangent line has slope -1.

(a) Differentiate implicitly:

$$2x + 7y + 7xy' + 2yy' = 0 \quad (*)$$

$$\Rightarrow \quad y' = \frac{-2x - 7y}{7x + 2y}, \quad \text{so} \quad y'\big|_{(1,1)} = \frac{-9}{9} = -1 \ //.$$

Diff. (*) implicitly:

$$2 + 7y' + 7y' + 7xy'' + 2y'y' + 2yy'' = 0$$

At $(1,1)$: $x = 1$, $y = 1$, and $y' = -1$ (from above). Substituting:

$$2 - 7 - 7 + 7y'' + 2(-1)^2 + 2y'' = 0$$

$$\Rightarrow \quad y''\big|_{(1,1)} = \frac{10}{9} \ //$$

(b) Want: (i) $(x, y)$ on the hyperbola

(ii) $y' = \dfrac{-2x - 7y}{7x + 2y} = -1 \Rightarrow -2x - 7y = -7x - 2y \Rightarrow x = y$

To have (i), plug in $y = x$ into the eqn of the hyperbola:

$$x^2 + 7x^2 + x^2 = 9 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

So, the points are $(1, 1)$ and $(-1, -1)$.

**Problem 5 (8 points)**

Let $f(x) = x^{x^n}$ where $n$ is a positive integer, defined on $(0, \infty)$.

(a) Prove that $\lim_{x \to \infty} f(x) = \infty$.

(b) It can be shown that $f$ is strictly decreasing on $(0, a_n]$ and strictly increasing on $[a_n, \infty)$. Determine the value of $a_n$.

(c) Now set $n = 2$, i.e., $f(x) = x^{x^2}$, and let $f^{-1}$ denote the inverse of $f$ on $[a_2, \infty)$. Calculate $[f^{-1}]'(16)$.

(a) Note: $x^{x^n} > 2^{x^n} \quad \forall x > 2$

$2^{x^n} > 2^x \quad \forall x > 1 \quad (\text{and } \forall n \in \mathbb{N})$

$\Rightarrow \quad x^{x^n} > 2^x \quad \forall x > 2 \quad (*)$

As $\lim_{x \to \infty} 2^x = \infty$, $(*)$ implies that $\lim_{x \to \infty} x^{x^n} = \infty$.

(b) $y = x^{x^n}$; $\ln y = x^n \ln x = x^n \ln x$

$\Rightarrow \dfrac{y'}{y} = n x^{n-1} \ln x + x^n \cdot \dfrac{1}{x} = x^{n-1}(n \ln x + 1)$

$\Rightarrow \quad y' = x^{x^n} \cdot x^{n-1}(n \ln x + 1)$

Since $x^{x^n + n - 1} > 0 \quad \forall x > 0$,

$f'(x) = 0 \iff n \ln x + 1 = 0 \iff \ln x = -\frac{1}{n} \Rightarrow x = e^{-1/n}$.

$f'(x) > 0 \iff n \ln x + 1 > 0 \iff \ln x > -\frac{1}{n} \iff x > e^{-1/n}$

$f'(x) < 0 \iff x < e^{-1/n}$

So, we conclude that $f$ is strictly increasing on $[e^{-1/n}, \infty)$ and strictly decreasing on $(0, e^{-1/n})$. That is $a_n = e^{-1/n}$

(c) $\cancel{\text{III}}$ $(f^{-1})'(16) = \dfrac{1}{f'(f^{-1}(16))}$

* $u = f^{-1}(16) \iff f(u) = 16 \iff u^{u^2} = 16 \iff u = 2$

that is $f^{-1}(16) = 2$

* $f'(2) = 2^{2^2} \cdot 2^{2-1}(2\ln x + 1)$ (from part (b) $\cancel{\}}$ with $n=2$)

$= 2^4 \cdot 2(2\ln x + 1) = 32(2\ln x + 1)$

So: $(f^{-1})'(16) = \dfrac{1}{f'(2)} = \dfrac{1}{32(2\ln 2 + 1)}$ //