MIDTERM 1 - SOLUTIONS

Math 120 Midterm 1
Oct 9, 2013
Duration: 50 minutes

Name: ______________________ Student Number: ______________________

This exam should have 8 pages. No textbooks, calculators, or other aids are allowed. There are 4 questions in this exam: problems 1 and 2 are 10 points each, problems 3 and 4 are 15 points each.

Problem 1 (10 points)

Consider the function $f(x) = \sqrt{x^2 + 5}$.

(a) What is the domain of f?

(b) What is the range of f? Justify your answer.

(c) Find all solutions of the inequality

$$f(x) \geq \sqrt{x^4 - 1}.$$

(a) $\mathrm{dom}(f) = \mathbb{R}$ (as $x^2+5 \geq 5$ $\forall x \in \mathbb{R}$, so the square-root function is real-valued.)

(b) First, note that $x^2+5 \geq 5$ $\forall x \in \mathbb{R}$ $\Rightarrow \sqrt{x^2+5} \geq \sqrt{5}$ $\forall x \in \mathbb{R}$.

Thus, $\mathrm{ran}(f) \subseteq [\sqrt{5}, \infty)$. $(*)$

Next, let $u \in [\sqrt{5}, \infty)$. We will show that $\exists x_0 \in \mathbb{R}$ such that $f(x_0) = u$: indeed, $\sqrt{x^2+5} = u \Rightarrow x^2+5 = u^2$.

$\Rightarrow x^2 = u^2 - 5$. Since $u \geq \sqrt{5}$, $u^2 \geq 5 \Rightarrow u^2 - 5 \geq 0$,

so : $x_0 = \sqrt{u^2-5} \in \mathbb{R} = \mathrm{dom}(f)$.

This shows : $[\sqrt{5}, \infty) \subseteq \mathrm{ran}(f)$ $(**)$.

Finally, $(*)$ & $(**)$ can only be true if $\mathrm{ran}(f) = [\sqrt{5}, \infty)$.

(c) $\sqrt{x^2+5} \geq \sqrt{x^4-1}$. First note: $x^4-1 \geq 0 \Leftrightarrow (x^2-1)\underbrace{(x^2+1)}_{\geq 0\ \forall x} \geq 0$

$\Leftrightarrow (x-1)(x+1) \geq 0$

$\Leftrightarrow \boxed{x \in (-\infty, -1] \cup [1, \infty)}$ $(*)$

Next: square both sides

$x^2+5 \geq x^4-1$

$\Leftrightarrow x^4 - x^2 - 6 \leq 0 \quad \Leftrightarrow (x^2-3)\underbrace{(x^2+2)}_{\geq 0\ \forall x} \leq 0$

$\Leftrightarrow (x-\sqrt{3})(x+\sqrt{3}) \leq 0$

$\Leftrightarrow \boxed{x \in [-\sqrt{3}, \sqrt{3}]}$ $(**)$.

So, ~~the~~ x solves the inequality iff it satisfies both $(*)$ & $(**)$,

i.e., $\boxed{x \in [-\sqrt{3}, -1] \cup [1, \sqrt{3}]}$ $\leftarrow$ $\boxed{\text{final solution set}}$

Problem 2 (10 points)

Use the formal definition of limit to prove that

$$\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}.$$

First: $\frac{1}{x} - \frac{1}{3} = \frac{3-x}{3x}$.

Let $\varepsilon > 0$. Want to find $\delta > 0$ s.t. $0 < |x-3| < \delta \Rightarrow$

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| = \frac{|3-x|}{|3x|} < \varepsilon$$

Noting $0 < |x-3| < \delta \iff 3-\delta < x < 3+\delta$ & $x \neq 3$,

~~choose~~ let $\delta \leq 1 \Rightarrow 2 \leq x \leq 4$ $(x \neq 3)$.

Accordingly, if $0<|x-3|<\delta$ with $\delta < 1$

$$\frac{|3-x|}{3x} \leq \frac{\delta}{6}$$

So, if we choose ~~$\delta = \min\{1, \frac{\delta}{6}\}$~~

$\delta < \min\{1, 6\varepsilon\}$, we are done.

Problem 3 (15 points)

Evaluate the following if they exist (show your work, however no formal justification required).

(a) $\lim_{x\to 1} \frac{x^2-1}{x^4-1}$.

(b) $\lim_{x\to\infty} \frac{\sin^2 x}{x}$.

(c) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$.

(d) $f'(1)$ if $f(x) = (x^2+1)^3 \frac{1}{x^2}$.

(e) $\left.\frac{d}{dx}\sqrt{\frac{x+2}{x-1}}\right|_{x=2}$.

(a) $\lim_{x\to 1} \frac{x^2-1}{x^4-1} = \lim_{x\to 1} \frac{\cancel{x^2-1}}{(\cancel{x^2-1})(x^2+1)} = \frac{1}{2}$

(b) $0 \le \sin^2 x \le 1 \quad \forall x$

$\Rightarrow \quad 0 \le \frac{\sin^2 x}{x} \le \frac{1}{x} \quad \forall x.$

Then, the squeeze theorem along with the fact that $\lim_{x\to\infty} \frac{1}{x} = 0$ shows that $\lim_{x\to\infty} \frac{\sin^2 x}{x} = 0$.

(c) Recall: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$. Set $a = \sqrt[3]{x}$, $b = 1$.

$\Rightarrow \quad (\sqrt[3]{x}-1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1) = x - 1.$

Also $(\sqrt{x}-1)(\sqrt{x}+1) = x - 1.$

Then

$$\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \frac{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)(\sqrt{x}+1)}$$

$$= \frac{(x-1)(\sqrt{x}+1)}{(x-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}$$

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \frac{\sqrt{1}+1}{\sqrt[3]{1}+\sqrt[2]{1}+1} = \frac{2}{3}.$$

(d) $f'(x) = 3(x^2+1)^2 \cdot 2x \cdot \frac{1}{x^2} + (x^2+1)^3\left(-2\frac{1}{x^3}\right)$

$f'(1) = 3 \cdot 4 \cdot 2 \cdot 1 + 2^3 \cdot (-2) = 24 - 16 = 8$ //

(e) $$\frac{d}{dx}\sqrt{\frac{x+2}{x-1}} = \frac{1}{2\sqrt{\frac{x+2}{x-1}}} \cdot \frac{d}{dx}\frac{x+2}{x-1}$$

$$= \frac{1}{2\sqrt{\frac{x+2}{x-1}}} \cdot \frac{(x-1)-(x+2)}{(x-1)^2}.$$

$$\left.\frac{d}{dx}\sqrt{\frac{x+2}{x-1}}\right|_{x=2} = \frac{-3}{2\cdot\sqrt{4}\cdot 1^2} = -\frac{3}{4}$$

Problem 4 (15 points) Consider the function

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Prove that f is continuous at $x = 0$ (give a detailed proof).

(b) Use the formal definition of the derivative to show that $f'(0) = 0$.

(c) Note that we have

$$f'(x) = \begin{cases} 2x \cos \frac{1}{x} + \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is $f'(x)$ continuous on $\mathbb{R}$? Justify your answer.

(a) First, note that $0 \in \mathrm{dom}(f)$, i.e., $f(0)$ is defined.
Need to show: $\lim_{x \to 0} f(x) = 0$. This follows from the squeeze theorem:

$$-1 \leq \cos\left(\tfrac{1}{x}\right) \leq 1 \quad \forall x \in \mathbb{R}\setminus\{0\}.$$

$$\Rightarrow \; -x^2 \leq x^2 \cos\left(\tfrac{1}{x}\right) \leq x^2 \quad \forall x \in \mathbb{R}\setminus\{0\}. \; (\text{as } x^2 \geq 0)$$

Furthermore $\lim_{x \to 0} (-x^2) = 0$ & $\lim_{x \to 0} x^2 = 0$. By the squeeze theorem, this implies $\lim_{x \to 0} \cancel{\cos} \, x^2 \cos\left(\frac{1}{x}\right)$ must exist and $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$. Since $f(0) = 0$, this shows that f is continuous at $x = 0$.

(b) $$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \cos\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \cos\left(\tfrac{1}{h}\right).$$

Again, by squeeze theorem, noting that

$$-|h| \leq h \cos\left(\tfrac{1}{h}\right) \leq |h| \;\; \& \;\; \lim_{h \to 0} -|h| = \lim_{h \to 0} |h| = 0,$$

we conclude $f'(0) = 0$.

(c) If f' was cts at $x=0$, we'd have

$$\lim_{x\to 0} f'(x) = f'(0) = 0.$$

However, for $x \neq 0$, we are given that

$$f'(x) = 2x\cos\left(\tfrac{1}{x}\right) + \sin\left(\tfrac{1}{x}\right).$$

We have shown above (in the proof in (b)) that $x \cdot \cos\left(\frac{1}{x}\right)$ is cts at $x=0$.

So, if f' is also cts at $x=0$, this would imply that $\sin\left(\frac{1}{x}\right) = f'(x) - 2x\cos\left(\frac{1}{x}\right)$ is cts at $x=0$ (sums/differences of cts functions are cts) - which contradicts with the fact that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist (as proved in the homework). So, we conclude that $f'(x)$ is not cts at $x=0$.