QUESTION 1

(a) Solve ODE for K & F(t)

Set up differential equation

\[ 2x'' + Kx = F \]

Find

\[ x'' = -9c_1\cos(3t) - 9c_2\sin(3t) = e^t \]

Plug into differential equation

\[ (K - 18)[c_1\cos(3t) + c_2\sin(3t)] + (2 + K)e^t = F(t) \]

From equation of motion for undamped systems we know:

\[ \omega = \sqrt{\frac{K}{m}} \]

\[ 3 = \sqrt{\frac{K}{2}} \]

Therefore,

\[ K = 18 \]

Plug K=18 in the above Differential equation to get:

\[ F = 20e^t \]

(b) Find \( \mathcal{L}[\cos^2(\omega t)] \)

Trigonometric Identity

\[ \cos^2(\omega t) = \frac{1}{2} + \frac{\cos(2\omega t)}{2} \]

Then

\[ \mathcal{L}[\cos^2(\omega t)] = \mathcal{L}\left[\frac{1}{2}\right] + \mathcal{L}\left[\frac{\cos(2\omega t)}{2}\right] \]
Using the laplace transform tables we get
\[ \mathcal{L}[\cos^2(\omega t)] = \frac{1}{2s} + \frac{s}{2s^2 + 8\omega^2} \]

**QUESTION 2**

(i) Find solution to the homogenous equation

\[ t^2 y'' - 2y = 0 \]

Guess:

\[ y(t) = t^r \]

\[ y'(t) = rt^{r-1} \]

\[ y''(t) = r(r-1)t^{r-2} \]

Plug values of \( y(t) \) and \( y''(t) \) into the above equation:

\[ t^r(r^2 - r - 2) = 0 \]

Solve for \( r \):

\[ r_1 = -1, r_2 = 2 \]

Therefore, the solution to the homogenous equation is:

\[ y(t) = c_1 \frac{1}{t} + c_2 t^2 \]

(ii) Use variation of parameters to find a particular solution that solves

\[ y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2} \]

Assume

\[ y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \]
where
\[ y_1(t) = t^2, \quad y_2(t) = \frac{1}{t} \]

Find the wronskian:
\[
W(t) = \begin{vmatrix}
  t^2 & \frac{1}{t} \\
  2t & \frac{1}{t^2}
\end{vmatrix}
\]

\[ W(t) = -3 \]

Find \( u_1(t) \) & \( u_2(t) \) using the equation:
\[
u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)} \, dt
\]
\[
u_2(t) = \int \frac{y_1(t)g(t)}{W(t)} \, dt
\]

where \( W(t) = -3 \) & \( g(t) = 3 - \frac{1}{t^2} \)

After plugging in the values for \( W(t) \) and \( g(t) \) then integrating, we get:
\[
u_1(t) = \ln(t) + \frac{1}{6t^2}
\]
\[
u_2(t) = -\frac{t^3}{3} + \frac{t}{3}
\]

Using the assumed form of:
\[
y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)
\]

we find that:
\[
y_p(t) = t^2\ln(t) - \frac{t^2}{3} + \frac{1}{2}
\]

Combining the homogenous and particular solution, we get
\[
y(t) = c_1 \frac{1}{t} + c_2 t^2 + t^2\ln(t) + \frac{1}{2}
\]