HW9 solutions

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\[ \text{It's very possible these solutions contain mistakes or are not the most efficient.} \]

1. The proof provided in the solutions to HW8 only used that \( f \in C^1 \).

2. Fejér’s theorem states that for all \( f \in C^{2\pi} \), we have \( \sigma_n(f) \to f \) uniformly, where \( \sigma_n \) is the \( n \)th Cesàro sum. Recall that we have \( \sigma_n(f) = F_n \ast f \) from the last homework. To prove Fejér theorem, we will use the fact that \( \frac{1}{2\pi} \int_{-\pi}^\pi F_n(t) \, dt = 1 \) for all \( n \) and that \( \lim_{n \to \infty} \int_{|x| \leq \delta} F_n(t) \, dt = 1 \) for all \( \delta > 0 \). These statements have allegedly already been proven in class. Fix \( \epsilon > 0 \) and pick \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) if \( |x - y| < \delta \). This is possible as \( f \) is continuous on \([-\pi, \pi]\). Now pick \( N \) such that for all \( n > N \) we have \( \int_{|x| \leq \delta} F_n(t) \, dt > 1 - \epsilon \). For such \( n \) and any \( x \in [-\pi, \pi] \), we compute:

\[
|\sigma_n(x) - f(x)| = |f(x) - \int_{-\pi}^\pi f(x-t)F_n(t) \, dt| \\
\leq |f(x) - \int_{-\pi}^{\delta} f(x-t)F_n(t) \, dt| + |\int_{\delta}^{\pi} f(x-t)F_n(t) \, dt| + |\int_{-\pi}^{-\delta} f(x-t)F_n(t) \, dt| \\
\leq |f(x) - \int_{-\pi}^{\delta} (f(x-t) - f(x))F_n(t) \, dt| + |\int_{\delta}^{\pi} f(x)F_n(t) \, dt| + \|f\|_{\infty} \int_{-\pi}^{\pi} F_n(t) \, dt \\
+ \|f\|_{\infty} \int_{-\pi}^{\delta} F_n(t) \, dt \\
< |f(x) - f(x)| \int_{-\pi}^{\delta} F_n(t) \, dt + |\int_{\delta}^{\pi} (f(x-t) - f(x))F_n(t) \, dt| + \|f\|_{\infty} \epsilon + \|f\|_{\infty} \epsilon \\
< 4\epsilon \|f\|_{\infty},
\]

which shows that \( \sigma_n(f) \to f \) uniformly as \( \epsilon > 0 \) was arbitrary. As the \( \sigma_n(f) \) are trigonometric polynomials, this implies Weierstrass’ second theorem.

3. (a) For \( k \in \mathbb{N} \), let \( g_k = \text{sgn}(D_k) \), where \( D_k \) is the Dirichlet kernel. From the previous homework, we know

\[
s_j g(0) = g \ast D_j(0) \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) D_j(-t) \, dt \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{sgn}(D_j(t))(D_j(t)) \, dt \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_j(t)| \, dt.
\]
We also know that there exists some constant $c > 0$ such that $\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_j(t)| \, dt > c \log N$. We will aim to find some continuous $f_j$ such that

$$|s_j(g_j - f_j)(0)| \leq \frac{c}{2} \log j.$$  \tag{1}$$

This will be enough to show the result as then by reverse triangle inequality

$$|s_j f_j(0)| \geq |s_j g_j(0)| - |s_j(g_j - f_j)(0)| > \frac{c}{2} \log j. \tag{2}$$

Once (2) is obtained we may simply take $\tilde{f}_j = f_{m_j}$, where $m_j > e^{\frac{2j}{c}}$, so that $|s_{m_j} \tilde{f}_j(0)| > j$. We now show (1). We know $D_j(x)$ changes signs at a finite number of points, hence $g_j(x)$ has a finite number $\{x_1, ..., x_n\}$ of discontinuities. Let $\{a_k\}$ and $\{b_k\}$ and be a selection of points in $[-\pi, \pi]$ such that $a_i \in (x_{i-1}, x_i)$, $b_i \in (x_{i-1}, x_i)$, $b_i > a_i$ for all $i$ and furthermore, $\sum_{i=1}^{n} (a_i - b_{i-1}) < \frac{\log j}{\|4D_j\|_{\infty}}$. Now let $f_j$ be the piecewise linear function determined by $(-\pi, g_j(-\pi)), (b_0, g_j(b_0)), (a_1, g_j(a_1)), ..., (\pi, g_j(\pi))$. As $f_j$ and $g_j$ agree outside the intervals $[b_i, a_{i+1}]$, we have

$$|s_j(g_j - f_j)(0)| = \frac{1}{\pi} \int_{-\pi}^{\pi} (g_j - f_j)(t)|D_j(t)| \, dt$$

$$= \sum_{i=1}^{n} \int_{b_{i-1}}^{a_i} (g_j - f_j)(t)|D_j(t)| \, dt$$

$$\leq 2\|D_j\|_{\infty} \sum_{i=1}^{n} \int_{b_{i-1}}^{a_i} \, dt$$

$$< \frac{c}{2} \log j,$$

as desired.

(b) We will first show that the $f_n$ constructed in part a) can be chosen to be trigonometric polynomials. Suppose we have $|s_n g(0)| > 2(n + 1)$ for some $g \in C^{2\pi}$ with $\|g\|_{\infty} \leq 1$ and $n \in \mathbb{N}$. Note that this implies $|s_n \frac{g}{2}(0)| > n + 1$ and $\|\frac{g}{2}\| \leq \frac{1}{2}$. By Weierstrass’ second approximation theorem we may pick a trigonometric polynomial $T_n$ such that $\|T_n - \frac{g}{2}\| < \frac{1}{2(n + 1)}$. Observe that $\|T_n\|_{\infty} \leq \|T_n - \frac{g}{2}\|_{\infty} + \|\frac{g}{2}\|_{\infty} \leq 1$. Let $a_k$ denote the Fourier coefficients of $\frac{g}{2}$ and let $a'_k$ denote those of $T_n$. For $k \leq n$ we have

$$a_k - a'_k = -\frac{1}{\pi} \int_{-\pi}^{\pi} (\frac{g}{2} - T_n)(t) \cos(nt) \, dt$$

$$\leq 2\|\frac{g}{2} - T_n\|_{\infty}$$

$$< \frac{1}{n + 1}.$$ 

Thus,

$$|s_n T_n(0)| \geq |s_n \frac{g}{2}(0)| - |s_n (\frac{g}{2} - T_n)(0)|$$

$$>(n + 1) - (\frac{|a_0 - a'_0|}{2} + \sum_{k=1}^{n} |a_k - a'_k|)$$

$$\geq n,$$
as desired.

Now let \( \{f_n\} \) be trigonometric polynomials such that \(|s_n f_n(0)| > n \) and \( \|f_n\|_{\infty} \leq 1 \). WLOG these can be chosen such that their degrees \( d_n = \deg(f_n) \) are increasing. We select an increasing sequence \( \{M_n\} \) and decreasing sequence \( \{\epsilon_n\} \) satisfying the following requirements:

\[
M_n \geq d_n + 1 \quad \epsilon_n M_n \geq 2^n \quad \epsilon_n M_{n-1} \leq \frac{1}{2^n} \quad \epsilon_n \leq \frac{1}{2^n}.
\]

We define \( f(x) = \sum_{n=1}^{\infty} \epsilon_n f_{M_n}(x) \). By the Weierstrass M-test this will converge uniformly to a continuous function. To show the Fourier series of \( f \) diverges at 0 we will show \(|s_{M_n} f(0)|\) is unbounded. For \( n \geq 1 \), we have:

\[
|s_{M_n} f(0)| \geq \epsilon_n |s_{M_n} f_{M_n}(0)| - \sum_{k \neq n} \epsilon_k |s_{M_n} f_{M_k}(0)| \tag{3}
\]

By choice of \( \epsilon_n \), \( M_n \) and \( f_{M_n} \), we know \( \epsilon_n |s_{M_n} f_{M_n}(0)| \geq \epsilon_n M_n \geq 2^n \). Note that for a continuous function \( h \) with Fourier coefficients \( a_k \) and \( b_k \) we have \( a_k, b_k \leq \|h\|_{\infty} \). Hence, \(|s_n h(0)| = |\frac{a_0}{2} + \sum a_k| \leq n + 1 \). From this fact, and because \( f_{M_k} = s_{M_n} f_{M_k} \) for \( k < n \) by choice of \( M_n \):

\[
\sum_{k \neq n} \epsilon_k |s_{M_n} f_{M_k}(0)| = \sum_{k=1}^{n-1} \epsilon_k |s_{M_n} f_{M_k}(0)| + \sum_{n+1}^{\infty} \epsilon_k |s_{M_n} f_{M_k}(0)| \leq \sum_{k=1}^{n-1} 2^{-k} + \sum_{n+1}^{\infty} \epsilon_k (M_n + 1) \leq c,
\]

for some \( c \in \mathbb{R} \). From (3), this shows that \(|s_{M_n} f(0)| \geq 2^n - c \) and so \(|s_{M_n} f(0)| \) is unbounded.

(c) Let \( f \) be the continuous function constructed in part b). Recall that \( f_c(x) = f(x-c) \). Let \( \{r_i\} \) be an enumeration of the rationals in \([-\pi, \pi]\). We define \( g(x) = \sum_{1}^{\infty} \frac{f_{r_i}(x)}{2^i} \). By the Weierstrass M-test, this converges uniformly to a continuous function. Furthermore, by construction it’s \( 2\pi \)-periodic and its Fourier series converges on \( \mathbb{Q} \cap [-\pi, \pi] \).

(d) The divergence of the Fourier series of a given continuous function \( f \) does not contradict the fact that there exists some other sequence of trigonometric polynomials converging uniformly to \( f \).

4. (a) We know \( \text{sgn}(x) \) is odd and \( 2\pi \)-periodic, hence \( a_k = 0 \) for all \( k \geq 0 \). For \( b_k \):

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kt) \text{sgn}(t) \, dt
= \frac{1}{\pi} \left( \int_{-\pi}^{0} -\sin(kt) \, dt + \int_{0}^{\pi} \sin(kt) \, dt \right)
= \frac{1}{\pi} \left( \frac{1 - \cos(kt)}{k} + \frac{1 - \cos(k\pi)}{k} \right),
\]
which is equal to $\frac{1}{k\pi}$ for $k$ odd and equal to 0 for $k$ even. Hence, the Fourier series of $f$ is given by $f(x) = \frac{4}{\pi} \sum_{k=1}^\infty \frac{\sin(2(k-1)\pi)}{2k-1}$.

(b) We first show the trigonometric identity $\sum_{k=1}^n \cos((2k-1)t) = \frac{\sin(2nt)}{2\sin(t)}$. This may be seen by examining the geometric sum $\sum_{k=1}^n e^{it(2k-1)}$:

$$\sum_{k=1}^n e^{it(2k-1)} = e^{-it} \sum_{k=1}^n e^{2ikt}$$

$$= e^{-it} \frac{1 - e^{2i(n+1)t}}{1 - e^{2it}}$$

$$= e^{int} \frac{e^{-int} - e^{int}}{e^{-it} - e^{it}}$$

$$= e^{int} \frac{\sin(nt)}{\sin(t)}.$$

Taking the real part of $\sum_{k=1}^n e^{it(2k-1)} = e^{int} \frac{\sin(nt)}{\sin(t)}$ then gives the identity. For $s_n$ we thus have:

$$s_n(x) = \frac{4}{\pi} \sum_{k=1}^n \frac{\sin(2(n-1)x)}{2n-1}$$

$$= \frac{4}{\pi} \int_0^x \sum_{k=1}^n \cos(2(n-1)t) \, dt$$

$$= \frac{2}{\pi} \int_0^x \frac{\sin(2nt)}{\sin(t)} \, dt,$$

as desired.

(c) From b), we have $s_n'(x) = \frac{2\sin(2nx)}{\pi \sin(x)}$ and so $s(x)$ has critical points at $\frac{\pm k\pi}{2n}$, with $k \in \{1, ..., (n-1)\}$. By considering $s_n''(x)$, it can furthermore be verified that the local minimums of $s_n$ occur at $\frac{2\pi}{2n}, \frac{3\pi}{2n}, ..., \pi$ and the local maximums occur at $\frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, ..., -\pi$. We will first find the global max of $s_n$ on $[0, \pi]$. For $0 \leq k \leq n-1$, let $M_k$ be the value of $s_n$ occurring at $\frac{(2k+1)\pi}{2n}$. We wish to show $M_0 \geq M_k$ for $k \in \{1, ..., n-1\}$. For $2k+1 \leq n$ we have:

$$M_k - M_{k-1} = \int_{\frac{2k\pi}{2n}}^{\frac{(2k+1)\pi}{2n}} \frac{\sin(2nt)}{\sin(t)} \, dt$$

$$= \int_{\frac{2k\pi}{2n}}^{\frac{2k\pi}{2n}} \frac{\sin(2nx)}{\sin(t)} \, dt + \int_{\frac{2k\pi}{2n}}^{\frac{(2k+1)\pi}{2n}} \frac{\sin(2nt)}{\sin(t)} \, dt$$

$$< \int_{\frac{2k\pi}{2n}}^{\frac{(2k+1)\pi}{2n}} \frac{\sin(2nt)}{\sin(\frac{2k\pi}{2n})} \, dt + \int_{\frac{2k\pi}{2n}}^{\frac{(2k+1)\pi}{2n}} \frac{\sin(2nt)}{\sin(\frac{2\pi}{2n})} \, dt$$

$$= 0.$$
such $k$, by using the change of variables $u = \pi - x$, it can be verified that $M_k - M_{k-1} = -(M_{n-k-1} - M_{n-k})$. From this, and because $M_k - M_{k-1} < 0$ for $k$ such that $2k - 1 \leq n$, it follows that $s_n$ attains its maximum on $[0, \pi]$ at $x = \frac{\pi}{2n}$. A similar computation to the above gives us that $s_n(x)$ is non-negative on $[0, \pi]$. This means that the minimum of $s_n$ on $[0, \pi]$ is 0 as $s_n(0) = 0$. Because $s_n$ is an odd function, this implies $s_n(x) \leq 0$ for $x \in [-\pi, 0]$. Hence, the maximum of $s_n$ on $[-\pi, \pi]$ is achieved at $\frac{\pi}{2n}$.

(d) From $b)$ we may write:

$$s_n\left(\frac{\pi}{2n}\right) = \frac{2}{\pi} \int_0^{\pi/2n} \frac{\sin(2nt)}{\sin(t)} dt = \frac{2}{\pi} \int_0^{\pi/2n} \frac{\sin(t)}{\sin\left(\frac{t}{2n}\right)} \frac{1}{2n} dt = \frac{2}{\pi} \int_0^{\pi/2n} \frac{\sin(t)}{t} \frac{1}{\sin\left(\frac{t}{2n}\right)} dt.$$ 

Let $g_n(t) = \frac{\sin(t)}{t} \frac{1}{\sin\left(\frac{t}{2n}\right)}$ and $h(t) = \frac{\sin(t)}{t}$. Note that for any $t \in [0, \pi]$ we have $\frac{\sin(t)}{t} \xrightarrow{n \to \infty} 1$ from the fact $\lim_{x \to 0} \frac{x}{\sin(x)} = 1$. This shows that $g_n \to h$ pointwise. Furthermore, it can be verified that $V_0^\pi(g_n)$ can be uniformly bounded by some $K > 0$ for all $n$. Indeed, recall that for functions $g_1$ and $g_2$ of bounded variation we have $V(g_1 g_2) \leq V(g_1) \|g_2\|_\infty + V(g_2) \|g_1\|_\infty$. Hence,

$$V_0^\pi(g_n) \leq V_0^\pi\left(\frac{\sin(t)}{t}\right) \frac{1}{\sin\left(\frac{t}{2n}\right)} \|g_2\|_\infty + V_0^\pi\left(\frac{\sin\left(\frac{t}{2n}\right)}{\sin(t)}\right) \frac{\sin(t)}{t} \|g_1\|_\infty \leq \pi V_0^\pi\left(\frac{\sin(t)}{t}\right) + \pi.$$ 

Thus, from Helly’s second theorem (HW7) we have $\int_0^\pi \cos(\pi t) dt \to \int_0^\pi \cos(\pi t) dt$ and so by IBP we have

$$s_n\left(\frac{\pi}{2n}\right) \to \int_0^\pi \frac{\sin(t)}{t} dt.$$